

# Renormalization and wandering Jordan curves of rational maps <sup>\*</sup>

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## Abstract

We aim to decompose a connected Julia set of an iterated rational map into two completely invariant subsets whose components are compact such that the induced action on the non-trivial periodic components corresponds to renormalizations, and the action on the entire collection of components is encoded by a metric dendrite dynamics. We achieve this objective for post-critically finite rational maps with a stable Cantor multicurve: a completely invariant multicurve with a strictly increasing number of pullback curves in each of its homotopy class. Applying Thurston's characterization theorem, we provide concrete examples of rational maps with Cantor multicurves by a typical topological surgery.

## 1 Introduction

A rational map on the Riemann sphere generates a dynamical system by iteration. The sphere is canonically decomposed into the disjoint union of the Fatou set and the Julia set defined by whether the iterated sequence forms a normal family in a neighborhood of the point. It may happen that on a periodic subset of the Julia set the first return map behaves like another rational map on its own Julia set. The dynamics on the periodic subsets are usually called a renormalization.

Our main concern in the present work is to study how to detect a renormalization, and to describe how the grand orbit of the renormalized Julia set fits into the global Julia set. We will try to achieve this objective by decomposing the dynamical plane of a rational map along a stable multicurve.

For polynomials a lot has been done: If the filled-in Julia set is disconnected but is not a Cantor set and all the non-escaping critical points are contained in pre-periodic components, one can extract a stable multicurve from a finite pullback of an equipotential curve

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in the basin of infinity. This stable multicurve automatically induces a renormalization. The pioneering work of Branner-Hubbard on cubic polynomials [4] uses a tableau to prove that a Julia component is either a pullback copy of a renormalization or a point. DeMarco and Pilgrim then use tableau and an infinite tree to encode the location of these points and renormalization copies [11].

If the filled-in Julia set is connected and there are more than one periodic external rays landing at a common periodic point, then these periodic external rays cut the Julia set into pieces. Together with equipotential curves, these rays form 'puzzles' that play the role of multicurves. As before the dynamics on a periodic puzzle piece is a renormalization. And in various expanding cases the complement of the Julia set of the small renormalized copies consists of uncountable many point components. See for example [4, 8, 15, 27], among others.

Let us turn now to non-polynomial rational maps. If a sub-hyperbolic rational map has a disconnected Julia set, one can also extract a stable multicurve in the multiply-connected Fatou domains. This multicurve gives rise to a canonical decomposition of the Julia set. See for example [6, 20].

The situation of a map with a connected Julia set is much harder. In the present work we will concentrate on post-critically finite rational maps, i.e. rational maps whose critical orbits contain only finitely many points. For such a map, the Julia set is automatically connected.

There is a particular case of post-critically finite rational maps which have a well-known decomposition. This is the case when the map has a Jordan curve whose pre-image is again a single Jordan curve homotopic to itself relative to the post-critical set. Cutting along this curve will decompose the rational map (or its second iteration if necessary) into two polynomials. The rational map is called the mating of the two polynomials. See for example [23, 24]. However, the decomposed polynomials can not be considered as renormalizations in the usual sense, as in most cases none of the two small Julia sets is embedded into the original Julia set. The best one can get is a semi-conjugacy from the small Julia sets onto the original Julia set [22].

Our contribution in this work is to establish a new type of decomposition for post-critically finite rational maps, under the combinatorial assumption of the existence of a **Cantor multicurve**. Contrary to the equator of a mating, a Cantor multicurve is a multicurve whose consecutive pullbacks will generate a strictly increasing number of curves in each homotopy class. See §2 for the definition.

The stability of multicurves is only measured up to homotopy. It is thus impossible to literally 'cut' along a stable multicurve to obtain exact invariant pieces. The first and also the crucial step in our study is to promote a Cantor multicurve to a multi-annulus such that it is exactly invariant in certain sense. We will call such a dynamical system an **exact annular system** (see §3 for the definition). It will play the role of multiply-connected Fatou components in the disconnected case, and will allow us to decompose the Julia set into pairwise disjoint pieces.

Actually we were unable to construct the exact annular system directly from the Cantor multicurve. Instead, we will first modify the rational map to a branched covering

in its Thurston equivalence class such that it has a topological exact annular system. Then applying a theorem of Rees and Shishikura, we obtain a semi-conjugacy from the branched covering to the rational map. Finally we show that the exact annular system is preserved under the semi-conjugacy.

Our second step is to study the exact annular system. This is actually quite simple since it is expanding, as a sub-system of a post-critically finite rational map. We will prove that each component of its Julia set is a Jordan curve. As a byproduct, we get uncountably many wandering Jordan curves on the Julia set. Actually we will prove that for a post-critically finite rational map, the existence of a wandering Jordan curve on the Julia set is equivalent to the existence of a Cantor multicurve.

Consider the complement of the grand orbit of the Julia set of the annular system. We will prove that each component is either a single point, or an eventually periodic continuum. Moreover, the first return map on a periodic continuum will form a renormalization. Furthermore, the configuration of these components together with the grand orbit of the Julia set of the annular system, as well as their dynamical information will be completely encoded by an expanding dynamical system on a dendrite.

Our final task is to provide concrete examples of post-critically finite rational maps which admit a Cantor multicurve. It is not obvious that such maps should ever exist. We will construct a branched covering from a polynomial via a topological surgery such that it has a Cantor multicurve consisting of a single curve. We will show that under certain conditions the branched covering is Thurston equivalent to a rational map. Consequently, this rational map has a Cantor multicurve and hence has wandering Jordan curves on the Julia set. As far as we know, this type of wandering continua was only found before as components of disconnected Julia sets of rational maps [17]. It is known that a polynomial without irrational indifferent cycles contains no wandering continua on its Julia set if its Julia set is connected and locally connected, see [3, 12, 26]. This construction has a somewhat independent interest, as it provides a rich family of rational maps with prescribed renormalizations.

**Notations and definitions.** The following notations and definitions will be used throughout this paper.

- Denote by  $U \Subset V$  if  $\overline{U} \subset V$  for open sets  $U, V \subset \widehat{\mathbb{C}}$ , where  $\widehat{\mathbb{C}}$  is the Riemann sphere.
- Let  $A \subset \widehat{\mathbb{C}}$  be an annulus and  $E \subset A$  be a connected open or closed set. We say that  $E$  is contained in  $A$  **essentially** if  $E$  separates the boundary  $\partial A$ .
- A continuum  $E \subset \widehat{\mathbb{C}}$  is called  **$n$ -connected** with  $n \in \mathbb{N} \cup \{\infty\}$  if  $\widehat{\mathbb{C}} \setminus E$  has exactly  $n$  components.
- Let  $f$  be a rational map. Denote by  $\mathcal{J}_f$  the **Julia set** of  $f$  and  $\mathcal{F}_f$  the **Fatou set** of  $f$ . Refer to [2, 5, 18, 19] for the definitions and basic properties. The **post-critical set** of  $f$  is denoted by  $\mathcal{P}_f$ , refer to §2.1 for its definition. A connected subset  $E \subset \widehat{\mathbb{C}}$  is called **of simple type** (w.r.t.  $\mathcal{P}_f$ ) if there exists either a simply-connected domain  $U \subset \widehat{\mathbb{C}}$  such that  $E \subset U$  and  $U$  contains at most one point in  $\mathcal{P}_f$ , or an annulus  $A \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$  such that  $E \subset A$ ; and **of complex type** (w.r.t.  $\mathcal{P}_f$ ) otherwise.
- The **Julia set** of an annular system  $g$  is denoted by  $\mathcal{J}_g$ , refer to §3.1 for its definition.

**Main results.** Here are the main theorems that we will prove:

**Theorem 1.1.** *A post-critically finite rational map  $f$  with a Cantor multicurve  $\Gamma$  has an*

exact annular sub-system homotopic to  $\Gamma$  rel  $\mathcal{P}_f$ .  $\mathcal{J}_g \subset \mathcal{J}_f$  and each component of  $\mathcal{J}_g$  is a Jordan curve. In particular,  $\mathcal{J}_g$  contains a wandering Jordan curve.

**Theorem 1.2.** *Let  $f$  be a post-critically finite rational map. If  $K$  is a wandering continuum and is not 1-connected, then  $K$  is a Jordan curve and  $f$  has a Cantor multicurve.*

Let  $f$  be a post-critically rational map with a stable Cantor multicurve  $\Gamma$ . Denote by  $\mathcal{J}(\Gamma)$  the union of the grand orbit of the Julia set of the annular sub-system derived from Theorem 1.1. Denote by  $\mathcal{K}(\Gamma) = \widehat{\mathbb{C}} \setminus \mathcal{J}(\Gamma)$ .

**Theorem 1.3.** *Every component of  $\mathcal{K}(\Gamma)$  is either a single point or an eventually periodic non-trivial continuum. Each periodic component of  $\mathcal{K}(\Gamma)$  is either the closure of a quasi-disk which is a periodic Fatou domain of  $f$ , or a complex type continuum which is the filled-in Julia set of a renormalization of the map  $f$ . Moreover  $\mathcal{K}(\Gamma)$  has  $\#\Gamma + 1$  complex type components.*

**Theorem 1.4.** *Let  $f$  be a post-critically rational map with a stable Cantor multicurve  $\Gamma$ . Then there exist an expanding finite dendrite map  $\tau : \mathcal{T} \rightarrow \mathcal{T}$  and a semi-conjugacy  $\Theta$  from  $f$  to  $\tau$  such that for each point  $t \in \mathcal{T}$ , the fiber  $\Theta^{-1}(t)$  is a component of  $\mathcal{J}(\Gamma)$  or  $\mathcal{K}(\Gamma)$ .*

Refer to §5 and §6 for the definitions of renormalization and finite dendrite maps.

This manuscript is organized as follows. In §2, we recall Thurston's theory and give the definition of Cantor multicurves and some equivalent conditions in the irreducible case. In §3, we introduce the notion of exact annular systems and show that every component of their Julia set is a Jordan curve if they are expanding. Theorem 1.1 is proved in §4. In §5, we will study the decomposition pieces and prove Theorem 1.3. In §6, we introduce the definition of finite dendrite maps and prove Theorem 1.4. Theorem 1.2 is proved in §7. In §8, we define a folding surgery on polynomials and prove that it provides branched coverings that are Thurston equivalent to rational maps under certain conditions.

## 2 Multicurves

In this section, we will recall Thurston's Theorem, introduce the notion of Cantor multicurves, and provide some equivalent conditions.

Let  $F$  be a branched covering of the Riemann sphere  $\widehat{\mathbb{C}}$ . We always assume  $\deg F \geq 2$  in this paper. Denote by  $\Omega_F$  the set of critical points of  $F$ . The **post-critical set** of  $F$  is defined by

$$\mathcal{P}_F = \overline{\bigcup_{n \geq 1} F^n(\Omega_F)}.$$

The map  $F$  is called **post-critically finite** if  $\mathcal{P}_F$  is finite.

A Jordan curve  $\gamma$  in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_F$  is **null-homotopic** (resp. **peripheral**) if one of its complementary components contains zero (resp. one) point of  $\mathcal{P}_F$ ; or is **essential** otherwise, i.e. if each of its two complementary components contains at least two points of  $\mathcal{P}_F$ .

A **multicurve**  $\Gamma$  is a non-empty and finite collection of disjoint Jordan curves in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_F$ , each essential and no two homotopic rel  $\mathcal{P}_F$ . We will say that  $\Gamma$  is **stable** if each essential curve in  $F^{-1}(\beta)$  for  $\beta \in \Gamma$  is homotopic rel  $\mathcal{P}_F$  to a curve in  $\Gamma$ ; and **pre-stable** if each curve  $\gamma \in \Gamma$  is homotopic rel  $\mathcal{P}_F$  to a curve in  $F^{-1}(\beta)$  for some curve  $\beta \in \Gamma$ . A pre-stable multicurve  $\Gamma$  is called **irreducible** if for each pair  $(\gamma, \beta) \in \Gamma \times \Gamma$ , there exists a sequence  $\{\delta_0 = \gamma, \delta_1, \dots, \delta_n = \beta\}$  of curves in  $\Gamma$  such that  $F^{-1}(\delta_k)$  has a component homotopic to  $\delta_{k-1}$  rel  $\mathcal{P}_F$  for  $1 \leq k \leq n$ .

Let  $\Gamma$  be a multicurve. Its **transition matrix**  $M_\Gamma = (a_{\gamma\beta})$  is defined by:

$$a_{\gamma\beta} = \sum_{\alpha} \frac{1}{\deg(f : \alpha \rightarrow \beta)},$$

where the summation is taken over components  $\alpha$  of  $F^{-1}(\beta)$  which are homotopic to  $\gamma$  rel  $\mathcal{P}_F$ . Denote by  $\lambda_\Gamma$  the leading eigenvalue of its transition matrix  $M_\Gamma$ . A stable multicurve  $\Gamma$  is called a **Thurston obstruction** if  $\lambda_\Gamma \geq 1$ .

Two post-critically finite branched coverings  $F$  and  $G$  are called **Thurston equivalent** if there is a pair of homeomorphisms  $(\phi, \psi) : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  such that  $\phi$  is isotopic to  $\psi$  rel  $\mathcal{P}_F$  and  $\phi \circ F \circ \psi^{-1} = G$ .

**Theorem 2.1. (Thurston Theorem)** *Let  $F$  be a post-critically finite branched covering of  $\widehat{\mathbb{C}}$  with hyperbolic orbifold. Then  $F$  is Thurston equivalent to a rational map  $f$  if and only if  $F$  has no Thurston obstruction. Moreover, the rational map  $f$  is unique up to holomorphic conjugation.*

Refer to [10] or [18] for the definition of hyperbolic orbifold.

**Lemma 2.2.** *Let  $F$  be a branched covering of  $\widehat{\mathbb{C}}$ . For any pre-stable multicurve  $\Gamma_0$ , there is a stable and pre-stable multicurve  $\Gamma$  such that  $\Gamma \supset \Gamma_0$  and hence  $\lambda_{\Gamma_0} \leq \lambda_\Gamma$ . Conversely, for any stable multicurve  $\Gamma$  with  $\lambda_\Gamma > 0$ , there is an irreducible multicurve  $\Gamma_0 \subset \Gamma$  such that  $\lambda_{\Gamma_0} = \lambda_\Gamma$ .*

Refer to [18] for the second part of the lemma. We only prove the first part.

*Proof.* Let  $\tilde{\Gamma}_n$  be the collection of essential curves in  $F^{-n}(\Gamma_0)$  for  $n \geq 1$ . Let  $\Gamma_n$  be a sub-collection of  $\tilde{\Gamma}_n$  such that no two curves in  $\Gamma_n$  are homotopic rel  $\mathcal{P}_F$  and any curve in  $\tilde{\Gamma}_n$  is homotopic rel  $\mathcal{P}_F$  to a curve in  $\Gamma_n$ . Then  $\Gamma_n$  is a pre-stable multicurve and each curve in  $\Gamma_n$  is homotopic to a curve in  $\Gamma_{n+1}$  for  $n \geq 1$ . Thus  $\#\Gamma_n \leq \#\Gamma_{n+1}$ . Since for any multicurve  $\Gamma$ ,  $\#\Gamma \leq \#\mathcal{P}_F - 3$ , there is an integer  $N \geq 0$  such that  $\#\Gamma_N = \#\Gamma_{N+1}$ . Thus  $\Gamma_N$  is a stable and pre-stable multicurve.  $\square$

**Convention.** Let  $\Gamma$  be a collection of curves in  $\widehat{\mathbb{C}}$ , we also use  $\Gamma$  to denote the union of curves in  $\Gamma$  as a subset of  $\widehat{\mathbb{C}}$  if there is no confusion.

Let  $\Gamma$  be a multicurve of  $F$ . For each  $\gamma \in \Gamma$ , denote by  $\Gamma(1, \gamma)$  the collection of curves in  $F^{-1}(\Gamma)$  homotopic rel  $\mathcal{P}_F$  to  $\gamma$  and  $\Gamma(1, \Gamma) := \bigcup_{\gamma \in \Gamma} \Gamma(1, \gamma)$ . Inductively, for  $n \geq 1$ , denote by  $\Gamma(n+1, \gamma)$  the collection of curves in  $F^{-1}(\Gamma(n, \Gamma))$  homotopic rel  $\mathcal{P}_F$  to  $\gamma$  and

$\Gamma(n+1, \Gamma) := \bigcup_{\gamma \in \Gamma} \Gamma(n+1, \gamma)$ . Notice that  $\Gamma(n, \Gamma)$  is contained in, but may not be equal to, the collection of curves in  $F^{-n}(\Gamma)$  homotopic rel  $\mathcal{P}_F$  to curves in  $\Gamma$ . Denote by

$$\kappa_n(\gamma) = \#\Gamma(n, \gamma) \text{ for each } \gamma \in \Gamma.$$

**Definition 1.** A multicurve  $\Gamma$  is called a **Cantor multicurve** if it is pre-stable and  $\kappa_n(\gamma) \rightarrow \infty$  as  $n \rightarrow \infty$  for all  $\gamma \in \Gamma$ .

A stable Cantor multicurve is in particular both stable and pre-stable. In the classical construction of mating of polynomials, there is a Jordan curve whose pre-image is a single curve homotopic to itself rel the post-critical set. In this case the multicurve consisting of this single curve is not a Cantor multicurve. It is quite easy to give examples of maps without Cantor multicurves, for instance topological polynomials (branched coverings with a totally invariant point). Concrete examples of rational maps with Cantor multicurves will be constructed in §8.

**Lemma 2.3.** Suppose that  $\Gamma$  is an irreducible multicurve. The following statements are equivalent:

- (1)  $\#\Gamma(1, \Gamma) > \#\Gamma$ .
- (2)  $\kappa_1(\gamma) \geq 2$  for some  $\gamma \in \Gamma$ .
- (3)  $\kappa_n(\gamma) \rightarrow \infty$  for some  $\gamma \in \Gamma$ .
- (4)  $\kappa_n(\gamma) \rightarrow \infty$  for all  $\gamma \in \Gamma$ , i.e.,  $\Gamma$  is a Cantor multicurve.
- (5) There is a curve  $\beta \in \Gamma$  such that  $F^{-1}(\beta)$  has at least two curves in  $\Gamma(1, \Gamma)$ .

*Proof.* (1)  $\iff$  (2): Since  $\Gamma$  is pre-stable,  $\Gamma(1, \gamma)$  is non-empty for each  $\gamma \in \Gamma$ . Thus  $\#\Gamma(1, \Gamma) > \#\Gamma$  if and only if  $\kappa_1(\gamma) \geq 2$  for some  $\gamma \in \Gamma$ .

(1)  $\iff$  (3): Since  $\Gamma$  is irreducible,  $F^{-1}(\gamma)$  has at least one curve contained in  $\Gamma(1, \Gamma)$  for each  $\gamma \in \Gamma$ . Thus if  $\#\Gamma(1, \Gamma) > \#\Gamma$ , then  $\#\Gamma(n+1, \Gamma) > \#\Gamma(n, \Gamma)$  for all  $n \geq 1$ . So  $\#\Gamma(n, \Gamma) \rightarrow \infty$  as  $n \rightarrow \infty$ . Therefore  $\kappa_n(\gamma) \rightarrow \infty$  for some  $\gamma \in \Gamma$ . Conversely, if  $\#\Gamma(1, \Gamma) = \#\Gamma$ , then  $\#\Gamma(n+1, \Gamma) = \#\Gamma(n, \Gamma)$  for all  $n \geq 1$ . Therefore  $\kappa_n(\gamma) = 1$  for all  $\gamma \in \Gamma$  and  $n \geq 1$ .

(3)  $\iff$  (4): Since  $\Gamma$  is irreducible, for each pair  $(\gamma, \beta) \in \Gamma \times \Gamma$ , there is an integer  $n \geq 1$  such that  $F^{-n}(\beta)$  has a component  $\delta$  homotopic to  $\gamma$  rel  $\mathcal{P}_F$  and  $F^k(\delta)$  is homotopic to a curve in  $\Gamma$  for  $1 \leq k < n$ . Therefore  $\delta \in \Gamma(n, \gamma)$  and hence  $\kappa_{n+k}(\gamma) \geq \kappa_n(\beta)$ . So  $\kappa_n(\gamma) \rightarrow \infty$  if  $\kappa_n(\beta) \rightarrow \infty$ .

(1)  $\iff$  (5): Since  $\Gamma$  is irreducible,  $F^{-1}(\gamma)$  has at least one curve contained in  $\Gamma(1, \Gamma)$  for each  $\gamma \in \Gamma$ . Therefore  $\#\Gamma(1, \Gamma) > \#\Gamma$  if and only if there is a curve  $\beta \in \Gamma$  such that  $F^{-1}(\beta)$  has at least two distinct curves contained in  $\Gamma(1, \Gamma)$ .  $\square$

Let  $\Gamma = \{\gamma_1, \dots, \gamma_n\}$  be a multicurve of  $F$ . Its **reduced transition matrix**  $M_{r, \Gamma} = (b_{ij})$  is defined by  $b_{ij} = k$  if there are  $k$  components of  $F^{-1}(\gamma_j)$  homotopic to  $\gamma_i$  rel  $\mathcal{P}_F$ . This definition was introduced by Shishikura.

**Lemma 2.4.** Let  $\Gamma$  be a pre-stable multicurve of  $F$ . Then the leading eigenvalue of its reduced transition matrix satisfies that  $\lambda(M_{r, \Gamma}) \geq 1$ . Moreover,  $\lambda(M_{r, \Gamma}) > 1$  if  $\Gamma$  is a Cantor multicurve. Conversely, if  $\Gamma$  is irreducible and  $\lambda(M_{r, \Gamma}) > 1$ , then  $\Gamma$  is a Cantor multicurve.



*Proof.* Note that  $M_{r,\Gamma}\mathbf{v} \geq \mathbf{v}$  for the vector  $\mathbf{v} = (1, \dots, 1)$  since  $\Gamma$  is pre-stable. Thus  $\lambda(M_{r,\Gamma}) \geq 1$  by Lemma A.1 in [6]. If  $\Gamma$  is a Cantor multicurve, then there exists an integer  $n \geq 1$  such that  $M_{r,\Gamma}^n \mathbf{v} \geq 2\mathbf{v}$ . Thus  $\lambda(M_{r,\Gamma})^n = \lambda(M_{r,\Gamma}^n) > 1$ . Conversely, if  $\Gamma$  is irreducible and  $\lambda(M_{r,\Gamma}) > 1$ , then there exists at least one column of the matrix such that the summation of the entries of this column is bigger than one. Thus  $\Gamma$  is a Cantor multicurve by Lemma 2.3 (2).  $\square$

### 3 Annular systems

**Definition 2.** By a **multi-annulus** we mean a finite disjoint union of open annuli in  $\widehat{\mathbb{C}}$  with finite modulus. Let  $\mathcal{A}^1, \mathcal{A} \subset \widehat{\mathbb{C}}$  be two multi-annuli such that each component of  $\mathcal{A}^1$  is contained in a component of  $\mathcal{A}$  essentially. A map  $g : \mathcal{A}^1 \rightarrow \mathcal{A}$  is called an **annular system** if

- (1) for each component  $A^1$  of  $\mathcal{A}^1$ , its image  $g(A^1)$  is a component of  $\mathcal{A}$  and the map  $g : A^1 \rightarrow g(A^1)$  is a holomorphic covering;
- (2) there is an integer  $n \geq 1$  such that for each component  $A$  of  $\mathcal{A}$ , the set  $g^{-n}(\mathcal{A}) \cap A$  is non-empty and disconnected.

The **Julia set** of  $g$  is defined by  $\mathcal{J}_g := \bigcap_{n \geq 0} g^{-n}(\mathcal{A})$ . An annular system  $g : \mathcal{A}^1 \rightarrow \mathcal{A}$  is called **proper** if  $\mathcal{A}^1 \Subset \mathcal{A}$ ; or **exact** if for every component  $A$  of  $\mathcal{A}$ , each of the two components of  $\partial A$  is also a component of  $\partial(A \cap \mathcal{A}^1)$ .

**Remark.** The definition of the Julia set  $\mathcal{J}_g$  of an annular system  $g$  is misleading. At first,  $\mathcal{J}_g$  need not to be compact. Secondly,  $g^{-1}(\mathcal{J}_g) = \mathcal{J}_g$  and  $g(\mathcal{J}_g) \subset \mathcal{J}_g$  from the definition but  $g(\mathcal{J}_g)$  need not to be equal to  $\mathcal{J}_g$  since we do not require the map  $g$  to be onto.

#### 3.1 Basic properties

**Proposition 3.1.** Let  $g : \mathcal{A}^1 \rightarrow \mathcal{A}$  be an annular system. Then there is an integer  $N \geq 1$  such that  $\deg(g^N|_A) \geq 2$  for each component  $A$  of  $g^{-N}(\mathcal{A})$ .

*Proof.* Let  $m \geq 1$  be the number of components of  $\mathcal{A}$ . By contradiction we assume that there is a component  $A$  of  $g^{-m}(\mathcal{A})$  such that  $\deg(g^m|_A) = 1$ . Then there exist integers  $0 \leq k < k+p \leq m$  such that both  $g^k(A)$  and  $g^{k+p}(A)$  are contained in the same component  $A^0$  of  $\mathcal{A}$ . So  $g^p(g^k(A)) \subset A^0$ . Let  $A^p \subset A^0$  be the component of  $g^{-p}(A^0)$  containing  $g^k(A)$ . Since  $g^p : A^p \rightarrow A^0$  is a covering between annuli and  $g^k(A)$  is contained essentially in  $A^p$ , we have

$$\deg(g^p : A^p \rightarrow A^0) = \deg(g^p : g^k(A) \rightarrow g^{k+p}(A)) \leq \deg(g^m|_A) = 1.$$

Thus the moduli of the annuli  $A^p$  and  $A^0$  are equal and hence  $A^p = A^0$ . It follows that  $A^0 = g^p(A^0)$ . Therefore  $A^0 \cap g^{-np}(\mathcal{A}) = A^0$  for all integers  $n \geq 1$ . Since  $g^{-n}(\mathcal{A}) \subset g^{-n+1}(\mathcal{A})$  for all  $n \geq 1$ , we conclude that  $A^0 \cap g^{-n}(\mathcal{A}) = A^0$  for all  $n \geq 1$ . This contradicts the condition that  $A^0 \cap g^{-n}(\mathcal{A})$  is disconnected for some  $n \geq 1$ . So  $\deg(g^m|_A) \geq 2$  for each component  $A$  of  $g^{-m}(\mathcal{A})$ .  $\square$

**Proposition 3.2.** *Let  $g : \mathcal{A}^1 \rightarrow \mathcal{A}$  be an exact annular system. Let  $\{A^n\}$  be a nested sequence of annuli of  $g^{-n}(\mathcal{A})$ , i.e. the annulus  $A^n$  is a component of  $g^{-n}(\mathcal{A})$  and  $A^{n+1} \subset A^n$ . Then either  $\bigcap_{n \geq 1} A^n = \emptyset$  or for every  $n \geq 1$ , there is an integer  $m > n$  such that  $A^m \Subset A^n$ .*

*Proof.* Consider the nested sequence  $\{A^n\}$ . By exactness, either for any  $n \geq 1$ , there is an integer  $m > n$  such that  $A^m \Subset A^n$ , or there are an integer  $N \geq 1$  and a component  $L$  of  $\partial A^N$  such that  $L \subset \partial A^n$  for  $n \geq N$ .

We only need to show that  $\bigcap_{n \geq 1} A^n = \emptyset$  in the latter case. Since  $\mathcal{A}$  has only finitely many components, there are integers  $i > j > N$  such that  $g^i(A^i) = g^j(A^j)$ , denote by  $B^n = g^j(A^{n+j})$  for  $n \geq 0$ . Then  $\{B^n\}$  is a nested sequence of annuli of  $g^{-n}(\mathcal{A})$  which have a common boundary component  $L' = g^j(L)$ . Moreover,  $g^p(B^p) = B^0$  for  $p = i - j$ . It follows that  $g^p(B^{n+p}) = B^n$  for  $n \geq 0$ .

Note that  $B^p \neq B^0$ . Otherwise  $B^n = B^0$  for all  $n \geq 1$  and thus contradicts the condition that  $B^0 \cap g^{-n}(\mathcal{A})$  is disconnected for some  $n \geq 1$ . Therefore either  $g^p(L') = L'$  or  $g^p(L') \subset B^0$ . In both cases we have  $g^{2p}(L') = L'$ .

Let  $U$  be the component of  $\widehat{\mathbb{C}} \setminus L'$  containing  $B^0$  and  $\phi$  be a conformal map from  $U$  onto the unit disk  $\mathbb{D}$ . Then  $h := \phi \circ g^{2p} \circ \phi^{-1}$  is a holomorphic covering from  $\phi(B^{2p})$  to  $\phi(B^0)$ , which can be extended continuously to the unit circle. By the symmetric extension principle,  $h$  can be extended to a holomorphic covering map from the annulus  $V_1$  to  $V$ , where  $V_1, V$  are the unions of  $\phi(B^{2p}), \phi(B^0)$  with its reflection and the unit circle, respectively. Since  $V_1 \Subset V$ ,  $h$  is expanding under the hyperbolic metric of  $V$ . So  $\bigcap_{n > 0} h^{-n}(V) = \partial \mathbb{D}$  and hence  $\bigcap_{n > 0} h^{-n}(\phi(B^0)) = \emptyset$ . Note that  $\phi(B^{2np}) = h^{-n}(\phi(B^0))$ . Therefore  $\bigcap_{n > 0} B^{2np} = \emptyset$  and hence  $\bigcap_{n > 0} A^n = \emptyset$ .  $\square$

Let  $g : \mathcal{A}^1 \rightarrow \mathcal{A}$  be an annular system and  $K$  be a connected component of  $\mathcal{J}_g$ . Then for each  $n \geq 0$ , there is a component of  $g^{-n}(\mathcal{A})$ , denoted by  $A^n(K)$ , such that  $K \subset A^n(K)$ . Consequently,  $K \subset \bigcap_{n \geq 1} A^n(K)$ .

**Proposition 3.3.** *Let  $g : \mathcal{A}^1 \rightarrow \mathcal{A}$  be an exact annular system.*

- (1) *For any component  $K$  of  $\mathcal{J}_g$ ,  $K$  is a 2-connected continuum contained essentially in each  $A^n(K)$  and  $K = \bigcap_{n \geq 1} A^n(K)$ .*
- (2) *For each component  $A$  of  $\mathcal{A}$  and any point  $z \in A$ , there exist components  $K_1, K_2$  of  $\mathcal{J}_g$  in  $A$  such that the annulus bounded by  $K_1$  and  $K_2$  contains the point  $z$ .*

*Proof.* (1) For any  $n \geq 0$ , there is an integer  $m > n$  such that  $A^m(K) \Subset A^n(K)$  by Proposition 3.2. Since  $A^{n+1}(K)$  is contained essentially in  $A^n(K)$  for every  $n \geq 0$ ,  $\bigcap_{n \geq 0} A^n(K)$  is a 2-connected continuum contained essentially in each  $A^n(K)$ . By definition it is contained in  $\mathcal{J}_g$  and hence is equal to  $K$ .

(2) Let  $A_1^n, A_2^n \subset A$  be the components of  $g^{-n}(\mathcal{A})$  such that they share a common boundary component with  $A$ . Then  $\bigcap_{n \geq 0} (A_1^n \cup A_2^n) = \emptyset$  by Proposition 3.2. Thus there exists an integer  $m \geq 1$  such that  $z \notin (A_1^m \cup A_2^m)$ . Notice that there exists a component  $K_i$  of  $\mathcal{J}_g$  contained essentially in  $A_i^m$  ( $i = 1, 2$ ). Thus the annulus bounded by  $K_1$  and  $K_2$  contains the point  $z$ .  $\square$



By Proposition 3.3, for each component  $K$  of  $\mathcal{J}_g$ ,  $g(K)$  is a component of  $\mathcal{J}_g$  and each component of  $g^{-1}(K)$  is also a component of  $\mathcal{J}_g$ . We will say that a component  $K$  of  $\mathcal{J}_g$  is **periodic** if there is an integer  $p \geq 1$  such that  $g^p(K) = K$ ; or **pre-periodic** if  $f^k(K)$  is periodic for some integer  $k \geq 1$ ; or **wandering** otherwise.

**Proposition 3.4.** *Let  $g : \mathcal{A}^1 \rightarrow \mathcal{A}$  be an exact annular system. Then any pre-periodic component  $K$  of  $\mathcal{J}_g$  is a quasicircle.*

*Proof.* We only need to consider periodic components of  $\mathcal{J}_g$  since each component of their pre-images is also a quasicircle. Let  $K$  be a periodic component of  $\mathcal{J}_g$  with period  $p \geq 1$ . Then  $g^p(A^p(K)) = A^0(K)$  and  $A^p(K) \subseteq A^0(K)$  by Proposition 3.2. Now applying quasiconformal surgery, we have a quasiconformal map  $\phi$  of  $\widehat{\mathbb{C}}$  such that  $\phi \circ g^p \circ \phi^{-1} = z^d$  in a neighborhood of  $\phi(K)$ , where  $|d| = \deg(g^p|_{A^p(K)}) \geq 2$ . Thus  $K$  is a quasicircle.  $\square$

### 3.2 Semi-conjugacy to linear systems

Let  $g : \mathcal{A}^1 \rightarrow \mathcal{A}$  be an exact annular system. In this sub-section, we want to characterize the dynamics of the map  $g$  by a linear system as the following. Denote by  $A_1, \dots, A_n$  the components of  $\mathcal{A}$  and  $A_1^1, \dots, A_m^1$  the components of  $\mathcal{A}^1$ . Let

$$\mathcal{I} = I_1 \cup \dots \cup I_n \quad \text{and} \quad \mathcal{I}^1 = I_1^1 \cup \dots \cup I_m^1$$

be disjoint unions of closed intervals on  $\mathbb{R}^1$  such that

$$\mathcal{I}^1 \subset \mathcal{I}, \quad \partial \mathcal{I} \subset \partial \mathcal{I}^1 \text{ and } I_i^1 \subset I_j \text{ whenever } A_i^1 \subset A_j.$$

Let  $\sigma : \mathcal{I}^1 \rightarrow \mathcal{I}$  be a map such that  $\sigma(I_i^1) = I_j$  if  $g(A_i^1) = A_j$  and  $\sigma$  is linear on each  $I_i^1$ . Set

$$\mathcal{I}^n = \sigma^{-n}(\mathcal{I}) \text{ for } n \geq 0, \quad \mathcal{J}_\sigma = \bigcap_{n \geq 0} \mathcal{I}^n \text{ and } \mathcal{B}_\sigma = \bigcup_{n \geq 0} \partial \mathcal{I}^n.$$

Then  $\mathcal{B}_\sigma \subset \mathcal{J}_\sigma$ .

For any point  $x \in \mathcal{J}_\sigma$ , the **itinerary** of  $x$  is defined by  $i(x) = (j_0, j_1, \dots)$  if  $\sigma^k(x) \in I_{j_k}^1$ . Similarly, for each point  $z \in \mathcal{J}_g$ , define its **itinerary** by  $i_*(z) = (j_0, j_1, \dots)$  if  $g^k(z) \in A_{j_k}^1$ .

**Proposition 3.5.** *There is a continuous onto map  $\Pi : \mathcal{J}_g \rightarrow \mathcal{J}_\sigma \setminus \mathcal{B}_\sigma$  such that  $\sigma \circ \Pi = \Pi \circ g$  on  $\mathcal{J}_g$ . Moreover the following statements hold:*

- (1) *The linear system  $\sigma : \mathcal{I}^1 \rightarrow \mathcal{I}$  is expanding and  $\mathcal{J}_\sigma$  is a Cantor set.*
- (2) *For any two distinct points  $x, y \in \mathcal{J}_\sigma$ ,  $i(x) \neq i(y)$ .*
- (3)  *$\Pi^{-1}(x)$  is a component of  $\mathcal{J}_g$  for each point  $x \in \mathcal{J}_\sigma \setminus \mathcal{B}_\sigma$ .*

*Proof.* (1) To prove that the linear system  $\sigma : \mathcal{I}^1 \rightarrow \mathcal{I}$  is expanding, we only need to show that there is an integer  $n \geq 1$  such that for any  $x \in \mathcal{I}^n$ ,  $|(\sigma^n)'(x)| > 1$ . Set  $l_k, L_k$  to be the minimum and maximum of the length of the components of  $\mathcal{I}^k$ , respectively for each  $k \geq 0$ . Then  $L_{k+1} \leq L_k$  for any  $k \geq 1$ . To prove  $|(\sigma^n)'| > 1$ , it is sufficient to show that there is an integer  $n \geq 1$  such that  $L_n < l_0$ . Assume  $L_k \rightarrow L > 0$  as  $k \rightarrow \infty$ . Then for each  $k \geq 1$ , there is a component of  $\mathcal{I}^k$  whose length is at least  $L$ . Therefore, there exists a sequence  $\{I^k\}_{k \geq 1}$  with  $I^k$  a component of  $\mathcal{I}^k$ , such that  $I^k \supset I^{k+1}$  and  $|I^k| \geq L$ .

Set  $I^\infty = \bigcap_k I^k$ . Then  $|I^\infty| \geq L$  and  $|I^k| \rightarrow |I^\infty|$  as  $k \rightarrow \infty$ . In particular, there exists an integer  $k_0 \geq 0$  such that

$$\frac{|I^k|}{|I^\infty|} < \frac{L_1 + l_1}{L_1}$$

for  $k \geq k_0$ . Furthermore, there exists an integer  $k_1 \geq k_0$  such that  $I^{k_1}$  contains another component  $I$  of  $\mathcal{I}^{k_1+1}$  distinct from  $I^{k_1+1}$ . Thus

$$\frac{|I|}{|I^{k_1+1}|} \leq \frac{|I^{k_1}| - |I^{k_1+1}|}{|I^{k_1+1}|} < \frac{l_1}{L_1}.$$

Therefore

$$\frac{|\sigma^{k_1}(I)|}{|\sigma^{k_1}(I^{k_1+1})|} < \frac{l_1}{L_1}$$

since  $\sigma^{k_1}$  is linear on  $I^{k_1}$ . This is a contradiction.

Now each component of  $\mathcal{J}_\sigma$  is a single point since the linear system  $\sigma$  is expanding. For any  $x \in \mathcal{J}_\sigma$ , let  $I^k(x)$  be the component of  $\mathcal{I}^k$  containing  $x$ . If  $I^k(x)$  have common endpoint  $y$  for  $k$  large enough, then  $x = y$  and the other endpoint of  $I^k(x)$ , which is contained in  $\mathcal{J}_\sigma$ , converges to  $x$  as  $k \rightarrow \infty$ . Otherwise, both endpoints of  $I^k(x)$  are not the point  $x$  but converge to  $x$  as  $k \rightarrow \infty$ . So  $\mathcal{J}_\sigma$  is a perfect set hence a Cantor set.

(2) By the definition of itinerary, two points  $x, y \in \mathcal{J}_\sigma$  have same itinerary if and only if they are contained in the same component of  $\mathcal{J}_\sigma$ . Thus  $i(x) \neq i(y)$  if  $x \neq y$  by (1).

(3) Define a map  $\Pi : \mathcal{J}_g \rightarrow \mathcal{J}_\sigma$  by  $\Pi(z) = x$  if  $i_*(z) = i(x)$ . It is well-defined by (2). From Proposition 3.2, we have  $\Pi(\mathcal{J}_g) = \mathcal{J}_\sigma \setminus \mathcal{B}_\sigma$ .

It is easy to see that  $\sigma \circ \Pi = \Pi \circ g$  on  $\mathcal{J}_g$ , and  $\Pi^{-1}(x)$  is a component of  $\mathcal{J}_g$  for each point  $x \in \mathcal{J}_\sigma \setminus \mathcal{B}_\sigma$ . Fix any point  $x \in \mathcal{J}_\sigma \setminus \mathcal{B}_\sigma$ . The collection  $\{I^k(x) \cap \mathcal{J}_\sigma\}_{k \geq 1}$  forms a basis of neighborhoods of the point  $x$  in  $\mathcal{J}_\sigma \setminus \mathcal{B}_\sigma$ . Now  $\Pi^{-1}(\{I^k(x) \cap \mathcal{J}_\sigma\}) = A^k(\Pi^{-1}(x)) \cap \mathcal{J}_g$  is open in  $\mathcal{J}_g$  for every  $k \geq 1$ . So  $\Pi$  is continuous.  $\square$

Since both  $\mathcal{B}_\sigma$  and the set of pre-periodic points are countable sets, we have:

**Corollary 3.6.** *There are uncountably many wandering components in  $\mathcal{J}_g$ .*

For any point  $x \in \mathcal{J}_\sigma$ , its  $\omega$ -**limit set**  $\omega(x)$  is defined to be the set of points  $y \in \mathcal{J}_\sigma$  such that  $\sigma^{k_n}(x)$  converges to  $y$  as  $n \rightarrow \infty$  for a subsequence  $k_n \rightarrow \infty$ .

**Proposition 3.7.** *Let  $x \in \mathcal{J}_\sigma$  be a wandering point. Then  $\omega(x)$  is an infinite set.*

*Proof.* Assume that  $\omega(x)$  is finite. Define  $d(y_1, y_2)$  to be the Euclidean distance if  $y_1, y_2$  are contained in the same component of  $\mathcal{I}$ , or infinity otherwise. Then there exists a constant  $\delta > 0$  such that  $d(y_1, y_2) > \delta$  for any two distinct points  $y_1, y_2 \in \omega(x)$  and  $d(y_1, y_2) > \delta$  if  $y_1, y_2$  are contained in different components of  $\mathcal{I}^1$ . Take a constant  $M \in (1, \infty)$  such that  $|\sigma'(x)| < M$  for any point  $x \in \mathcal{I}^1$ . By the definition of  $\omega(x)$ , there exists a constant  $N \geq 1$  such that for any  $n \geq N$ , there exists a unique point  $y_n \in \omega(x)$  such that  $d(\sigma^n(x), y_n) < \delta/(2M)$ . Thus  $d(\sigma^{n+1}(x), \sigma(y_n)) < \delta/2$ . It follows that  $y_{n+1} = \sigma(y_n)$  for  $n \geq N$ . This contradicts the fact that  $\sigma$  is expanding.  $\square$

### 3.3 Local connectivity

Recall that each component of the Julia set of an exact annular system is a 2-connected continuum by Proposition 3.3.

**Theorem 3.8.** *Let  $g : \mathcal{A}^1 \rightarrow \mathcal{A}$  be an exact annular system and  $K$  be a component of  $\mathcal{J}_g$ . Let  $U$  and  $V$  be the two components of  $\widehat{\mathbb{C}} \setminus K$ . Then  $\partial U = \partial V = K$ .*

*Proof.* Assume that each component of  $\mathcal{A}$  contains at least two components of  $\mathcal{A}^1$  (otherwise we consider  $g^n$  for some  $n \geq 2$  by the definition). Then  $\|g'\| > 1$  under the hyperbolic metric of  $\mathcal{A}$ .

If  $K$  is eventually periodic, then  $K$  is a quasicircle by Proposition 3.4 and hence the theorem holds. Now we suppose that  $K$  is wandering. Let  $\Pi$  be a semi-conjugacy from  $g : \mathcal{J}_g \rightarrow \mathcal{J}_g$  to a linear system  $\sigma : \mathcal{I}^1 \rightarrow \mathcal{I}$  defined in Proposition 3.5. Then  $x = \Pi(K)$  is a wandering point. Thus  $\omega(x)$  is an infinite set by Proposition 3.7. In particular  $\omega(x)$  contains a point  $y \in \mathcal{I}$  such that  $y \notin \partial \mathcal{I}$ . It follows that there exists a component  $I^m$  of  $\sigma^{-m}(\mathcal{I})$  such that  $y \in I^m$  and  $I^m$  is contained in the interior of  $\mathcal{I}$ . Hence there exists an increasing sequence  $\{n_k\}_{k \geq 1}$  of positive integers such that  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$  and  $\sigma^{n_k}(x) \in I^m$ .

Denote by  $A^m$  the component of  $g^{-m}(\mathcal{A})$  corresponding to the interval  $I^m$ . Then  $A^m \Subset \mathcal{A}$  and  $g^{n_k}(K) \subset A^m$ . For any component  $J$  of  $\mathcal{J}_g$ , denote by  $A^n(J)$  the component of  $g^{-n}(\mathcal{A})$  that contains  $J$ . Then we have

$$g^{n_k}(A^{m+n_k}(K)) = A^m(g^{n_k}(K)) = A^m.$$

For each annulus  $W \Subset \mathcal{A}$ , define

$$\text{width}(W) = \sup_{z \in W} \left\{ d_W(z, \partial_+ W) + d_W(z, \partial_- W) \right\},$$

where  $\partial_\pm W$  denotes the two boundary components of  $W$  and  $d_W(z, \partial_\pm W)$  denotes the infimum of the length of arcs connecting  $z$  to  $\partial_\pm W$  in  $W$  under the hyperbolic metric of  $\mathcal{A}$ .

Pick an annulus  $W_0$  bounded by smooth curves such that  $W_0 \Subset \mathcal{A}$  and  $A^m \subset W_0$ . Then  $\text{width}(W_0) < \infty$  and there exists a constant  $\lambda > 1$  such that  $\|g'(z)\| \geq \lambda > 1$  for every point  $z \in g^{-1}(W_0)$ .

Denote by  $W_k$  the component of  $g^{-n_k}(W_0)$  that contains  $K$ . Then

$$A^{m+n_k-n_j}(g^{n_j}(K)) \subset g^{n_j}(W_k) \Subset A^{n_k-n_j}(g^{n_j}(K))$$

for  $0 \leq j \leq k$  (set  $n_0 = 0$ ). Note that  $A^{n_k-n_j}(g^{n_j}(K)) \subset W_0$  if  $n_k - n_j \geq m$ . Thus  $g^{n_j}(W_k) \subset W_0$  if  $k - j \geq m$ . Therefore  $\|(g^{n_k})'(z)\| \geq \lambda^{k-m}$  for any point  $z \in W_k$  since the finite orbit  $\{z, g(z), \dots, g^{n_k-1}(z)\}$  passes at least  $k - m$  times through the set  $g^{-1}(W_0)$  where  $\|g'\| \geq \lambda$  and  $\|g'\| > 1$  always. So

$$\text{width}(W_k) \leq \lambda^{m-k} \text{width}(W_0).$$

Hence  $\text{width}(W_k) \rightarrow 0$  as  $k \rightarrow \infty$ .

Clearly  $\partial U \cup \partial V \subset K$ . In order to prove that  $\partial U = \partial V = K$  we only need to show that  $K \subset \partial U$  by symmetry. Otherwise, assume  $z \in K \setminus \partial U$ . Then the spherical distance  $d(z, \partial U) > 0$ . Label the boundary components of  $W_k$  by  $\partial_{\pm} W_k$  such that  $\partial_+ W_k \subset U$ . Then  $d(z, \partial_+ W_k) > d(z, \partial U) > 0$ . Note that there exists a constant  $M > 0$  such that  $d_{W_k}(z, \partial_+ W_k) \geq M \cdot d(z, \partial_+ W_k)$  for all  $k \geq 0$ . Therefore

$$\text{width}(W_k) \geq d_{W_k}(z, \partial_+ W_k) \geq M \cdot d_U(z, \partial_{\pm} U) > 0.$$

This contradicts the fact that  $\text{width}(W_k) \rightarrow 0$  as  $k \rightarrow \infty$ .  $\square$

In the appendix we will give an example constructed by X. Buff showing that an exact annular system may have a non-locally connected wandering Julia component. The next theorem gives a sufficient condition about the local connectivity of wandering components. The idea of the proof comes from [20].

**Theorem 3.9.** *Let  $g : \mathcal{A}^1 \rightarrow \mathcal{A}$  be an exact annular system. Suppose that  $g$  is expanding, i.e. there exists a smooth metric  $\rho$  on  $\mathcal{A}$  and a constant  $\lambda > 1$  such that  $\|g'\| \geq \lambda$ . Then every component of  $\mathcal{J}_g$  is a Jordan curve.*

*Proof.* Pick a pre-periodic component of  $\mathcal{J}_g$  in each component of  $\mathcal{A}$  and denote by  $\Gamma_0$  the collection of them. It is a multicurve consisting of quasicircles. Denote by  $\Gamma_n$  the collection of curves in  $g^{-n}(\Gamma_0)$ . Then for any curve  $\gamma \in \Gamma_n$  and any curve  $\beta \in \Gamma_m$  with  $m \neq n$ , either they are disjoint or one coincides with another.

For each curve  $\beta \in \Gamma_1$ , there is a unique curve  $\gamma \in \Gamma_0$  such that  $\beta$  and  $\gamma$  are contained in the same component of  $\mathcal{A}$ . If  $\beta \neq \gamma$ , there is a homotopy  $\Phi_\beta : S^1 \times [0, 1] \rightarrow \mathcal{A}$  from  $\gamma$  to  $\beta$  such that  $\phi_t := \Phi_\beta(\cdot, t)$  is a homeomorphism for any  $t \in [0, 1]$ , and in particular,  $\phi_0(S^1) = \gamma$ ,  $\phi_1(S^1) = \beta$  and  $\phi_t(S^1)$  is a curve between  $\beta$  and  $\gamma$ . If  $\beta = \gamma$ , define  $\Phi_\beta(\cdot, t) : S^1 \rightarrow \beta$  to be a homeomorphism independent on  $t$ .

Define the homotopic length of a path  $\delta : [0, 1] \rightarrow \mathcal{A}$  by

$$\text{h-length}(\delta) = \inf \left\{ \text{length of } \alpha \text{ under metric } \rho \right\},$$

where the infimum is taken over all the path  $\alpha$  in  $\mathcal{A}$  connecting  $\delta(0)$  to  $\delta(1)$  and homotopic to  $\delta$ . Then

$$\text{h-length}(\tilde{\delta}) \leq \frac{1}{\lambda} \cdot \text{h-length}(\delta)$$

for any lift  $\tilde{\delta}$  of  $\delta$  under the map  $g$  since  $\|g'\| \geq \lambda$ .

For each  $\beta \in \Gamma_1$  and any  $s \in S^1$ ,  $\Phi_\beta(s, \cdot)$  maps the interval  $[0, 1]$  to a path  $\delta_{\beta, s}$  in the closed annulus  $\Phi_\beta(S^1 \times [0, 1])$  which connects two points in each of its boundary. So there is a constant  $C > 0$  such that  $\text{h-length}(\delta_{\beta, s}) < C$  for each  $\beta \in \Gamma_1$  and any  $s \in S^1$ .

For each wandering component  $K$  of  $\mathcal{J}_g$ , let  $\alpha_n$  be the unique curve of  $\Gamma_n$  with  $\alpha_n \subset A^n(K)$ . Then  $g^n(\alpha_n) \in \Gamma_0$  and  $\beta := g^n(\alpha_{n+1}) \in \Gamma_1$  are contained in the same component of  $\mathcal{A}$ . Now the homotopy  $\Phi_\beta$  from  $g^n(\alpha_n)$  to  $\beta$  defined above can be lifted to an homotopy

from  $\alpha_n$  to  $\alpha_{n+1}$ , denote it by  $\Psi_n : S^1 \times [0, 1] \rightarrow A^n(K)$ , by the following commutative diagram:

$$\begin{array}{ccc} S^1 \times [0, 1] & \xrightarrow{\Psi_n} & A^n(K) \\ P_d \downarrow & & \downarrow g^n \\ S^1 \times [0, 1] & \xrightarrow[\Phi_\beta]{} & g^n(A^n(K)), \end{array}$$

where  $d = \deg(g|_{A^n(K)})$  and  $P_d(s, t) = (s^d, t)$ , i.e.  $P(\cdot, t)$  is a covering of  $S^1$  with degree  $d$ . Set  $\psi_{n,t} := \Psi_n(\cdot, t)$ . It is a homeomorphism for any  $t \in [0, 1]$ , and in particular,  $\psi_{n,0}(S^1) = \alpha_n$  and  $\psi_{n,1}(S^1) = \alpha_{n+1}$ . For any  $s \in S^1$ ,  $\Psi_n(s, t)(S^1)$  is a path connecting a point in  $\alpha_n$  with a point in  $\alpha_{n+1}$  whose homotopic length is less than  $C\lambda^{-n}$ .

These isotopies  $\Psi_n$  can be pasted together to a continuous map  $\Psi : S^1 \times [0, \infty) \rightarrow \mathcal{A}$  as the following:

$$\Psi(s, t) = \begin{cases} \Psi_0(s, t) & \text{on } S^1 \times [0, 1] \\ \Psi_1(\psi_{1,0}^{-1} \circ \psi_{0,1}(s), t - 1) & \text{on } S^1 \times [1, 2] \\ \vdots & \vdots \\ \Psi_n(\psi_{n,0}^{-1} \circ \psi_{n-1,1} \circ \cdots \circ \psi_{1,0}^{-1} \circ \psi_{0,1}(s), t - n) & \text{on } S^1 \times [n, n+1] \\ \vdots & \vdots \end{cases}$$

Set  $h_t = \Psi(\cdot, t)$ . Then  $h_n(S^1) = \alpha_n$ . For each  $s \in S^1$  and any integers  $m > n \geq 0$ , the homotopic length of the path  $\zeta_s(n, m) := \{\Psi(s, t) : n \leq t \leq m\}$  satisfies:

$$\text{h-length} \left( \zeta_s(n, m) \right) \leq C\lambda^{-n} + \cdots + C\lambda^{1-m} \leq \frac{C}{(\lambda - 1)\lambda^{n-1}}.$$

Note that the two endpoints of  $\zeta_s(n, m)$  are  $h_n(s) \in \alpha_n$  and  $h_m(s) \in \alpha_m$ . The above inequality shows that  $\{h_n\}$  is a Cauchy sequence and hence converges uniformly to a continuous map  $h$ . Since  $\alpha_n \subset A^n(K)$ , we have  $h(S^1) \subset \bigcap_{n \geq 1} A^n(K) = K$ . Note that  $h(S^1)$  separates the two components of  $\widehat{\mathbb{C}} \setminus K$ . Thus  $h(S^1) = K$  by Theorem 3.8. Therefore  $K$  is locally connected and hence is a Jordan curve.  $\square$

## 4 From multicurves to annular systems

Let  $f$  be a post-critically finite rational map. In this section, we shall prove the next theorem which is a more precise version of Theorem 1.1.

**Convention.** We say that an annulus  $A \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$  is homotopic rel  $\mathcal{P}_f$  to a Jordan curve  $\gamma$  (or an annulus  $A'$ ) in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$  if essential Jordan curves in  $A$  are homotopic to  $\gamma$  (or essential curves in  $A'$ ) rel  $\mathcal{P}_f$ ; and a multi-annulus  $\mathcal{A}$  is homotopic rel  $\mathcal{P}_f$  to a multicurve  $\Gamma$  (or a multi-annulus  $\mathcal{A}'$ ) if each component of  $\mathcal{A}$  is homotopic to a curve in  $\Gamma$  (or a component of  $\mathcal{A}'$ ) rel  $\mathcal{P}_f$  and each curve in  $\Gamma$  (or each component of  $\mathcal{A}'$ ) is homotopic to a component of  $\mathcal{A}$ .

**Theorem 4.1.** *Let  $f$  be a post-critically finite rational map with a Cantor multicurve  $\Gamma$ . Then there is a unique multi-annulus  $\mathcal{A} \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$  homotopic rel  $\mathcal{P}_f$  to  $\Gamma$  such that  $g = f|_{\mathcal{A}^1} : \mathcal{A}^1 \rightarrow \mathcal{A}$  is an exact annular system, where  $\mathcal{A}^1$  is the union of components of  $f^{-1}(\mathcal{A})$  that are homotopic rel  $\mathcal{P}_f$  to curves in  $\Gamma$ . Moreover,  $\mathcal{J}_g \subset \mathcal{J}_f$  and each component of  $\mathcal{J}_g$  is a Jordan curve. In particular, there exists a wandering Jordan curve on  $\mathcal{J}_g$ .*

We were unable to prove this theorem directly. Instead we will take a detour to the space of branched coverings of the sphere. We will first modify topologically the rational map  $f$  to a branched covering  $F$  in its Thurston equivalence class such that  $F$  has a topological exact annular system. We then apply a theorem of Rees and Shishikura (refer to Theorem A.1 in the appendix) to obtain a semi-conjugacy from  $F$  to  $f$ . Finally we show that the existence of an exact annular system is preserved under the semi-conjugacy. Notice that we will not need Thurston Theorem for post-critically finite rational maps.

*Proof of Theorem 4.1.*

**Step 1. Topological modification.** Let  $f$  be a post-critically rational map with a Cantor multicurve  $\Gamma$ . Then there exists a multi-annulus  $\mathfrak{C} \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$  homotopic to  $\Gamma$  rel  $\mathcal{P}_f$  such that its boundary  $\partial\mathfrak{C}$  is a disjoint union of Jordan curves in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ . Let  $\mathfrak{C}^*$  be the union of all the components of  $f^{-1}(\mathfrak{C})$  which are homotopic to curves in  $\Gamma$  rel  $\mathcal{P}_f$ . Then for each  $\gamma \in \Gamma$ , there is at least one component of  $\mathfrak{C}^*$  homotopic to  $\gamma$  rel  $\mathcal{P}_f$  since  $\Gamma$  is pre-stable.

For each  $\gamma \in \Gamma$ , denote by  $\mathfrak{C}^*(\gamma)$  the smallest annulus containing all the components of  $\mathfrak{C}^*$  which are homotopic to  $\gamma$  rel  $\mathcal{P}_f$ . Then its boundary are two Jordan curves in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$  homotopic to  $\gamma$  rel  $\mathcal{P}_f$ . Set  $\mathfrak{C}^*(\Gamma) = \bigcup_{\gamma \in \Gamma} \mathfrak{C}^*(\gamma)$ . Then there exist a neighborhood  $U$  of  $\mathcal{P}_f$  and a homeomorphism  $\theta_0$  of  $\widehat{\mathbb{C}}$  such that  $\theta_0$  is isotopic to the identity rel  $U$  and  $\theta_0(\mathfrak{C}) = \mathfrak{C}^*(\Gamma)$ .

Set  $F := f \circ \theta_0$  and  $\mathfrak{C}^1 := \theta_0^{-1}(\mathfrak{C}^*)$ , then  $\mathcal{P}_F = \mathcal{P}_f$  and  $F$  is Thurston equivalent to  $f$  via the pair  $(\theta_0, \text{id})$ . Moreover, the restriction  $F|_{\mathfrak{C}^1} : \mathfrak{C}^1 \rightarrow \mathfrak{C}$  is a topological exact annular system.

**Step 2. Semi-conjugacy.** By Theorem A.1, there exist a neighborhood  $V$  of the critical cycles of  $F$  and a sequence  $\{\phi_n\}$  ( $n \geq 1$ ) of homeomorphisms of  $\widehat{\mathbb{C}}$  isotopic to the identity rel  $\mathcal{P}_F \cup V$  such that  $f \circ \phi_n = \phi_{n-1} \circ F$  and the sequence  $\{\phi_n\}$  converges uniformly to a continuous onto map  $h$  of  $\widehat{\mathbb{C}}$  and  $f \circ h = h \circ F$ .

**Step 3. Survival of the annular system.** This is the main step. Define

$$T = \{w \in \widehat{\mathbb{C}} : h^{-1}(w) \text{ crosses some component of } \mathfrak{C}\},$$

here we say a continuum  $E$  **crosses** an annulus  $C$  if  $E$  intersects with the both boundary components of  $C$ . Then  $T \subset \mathcal{J}_f$  by Theorem A.1 (3). It is easy to see that  $T$  is closed.

**Lemma 4.2.** *The set  $T$  is empty.*

This lemma is crucial. Here the property of Cantor multicurves is essential. It is not true for matings of polynomials. We prove at first a purely topological result.



**Lemma 4.3.** *Let  $\Gamma = \gamma_1 \cup \dots \cup \gamma_n$  be a finite disjoint union of Jordan curves on  $\widehat{\mathbb{C}}$  and  $L \subset \widehat{\mathbb{C}}$  be a continuum. Then for any Jordan domain  $D$  containing  $L$ , there is an integer  $N \geq 0$  such that for any two distinct points  $z_1, z_2 \in L$ , there exists a Jordan arc  $\delta$  in  $D$  connecting  $z_1$  with  $z_2$  such that  $\#(\delta \cap \Gamma) \leq N$ .*

*Proof.* Set

$$\Lambda = \{\alpha : \alpha \text{ is a component of } \Gamma \cap D \text{ such that } \alpha \cap L \neq \emptyset\}.$$

Then  $N := \#\Lambda < \infty$ . In fact, let  $\gamma : S^1 \times \{1, \dots, n\} \rightarrow \Gamma$  be a homeomorphism. Then  $\gamma^{-1}(\Gamma \cap L)$  is a compact subset, which is covered by the open intervals  $\{\gamma^{-1}(\alpha), \alpha \in \Lambda\}$ . Therefore  $\Lambda$  is finite.

For any two distinct points  $z_1, z_2 \in L$ , set

$$\Lambda(z_1, z_2) = \{\alpha \in \Lambda : \alpha \cup \partial D \text{ separates } z_1 \text{ from } z_2\}.$$

Then there exists a Jordan arc  $\delta \subset D$  connecting  $z_1$  with  $z_2$  such that  $\delta$  intersects with each  $\alpha \in \Lambda(z_1, z_2)$  on a single point and disjoint from other components of  $\Gamma \cap D$ . So  $\#(\delta \cap \Gamma) \leq \#\Lambda(z_1, z_2) \leq N$ .  $\square$

*Proof of Lemma 4.2.* Assume  $T \neq \emptyset$  by contradiction. Then  $f(T) \subset T$ . In fact, suppose  $w \in T$ , i.e.,  $h^{-1}(w)$  crosses some component of  $\mathfrak{C}$ , then  $h^{-1}(w)$  crosses some component  $C^1$  of  $\mathfrak{C}^1$ . By Theorem A.1 (4),  $h^{-1}(f(w)) = F(h^{-1}(w))$ . So  $h^{-1}(f(w))$  crosses  $F(C^1)$  which is a component of  $\mathfrak{C}$ , so  $f(w) \in T$ . Set  $T_\infty = \bigcap_{n \geq 0} f^n(T)$ . Then  $T_\infty$  is a non-empty closed set and  $f(T_\infty) = T_\infty$ .

Pick one point  $w_0 \in T_\infty$ . Since  $f(T_\infty) = T_\infty$ , there exists a sequence of points  $\{w_n\}_{n \geq 0}$  in  $T_\infty$  such that  $f(w_{n+1}) = w_n$  (i.e.  $T_\infty$  contains a backward orbit). Either  $w_n$  is periodic for all  $n \geq 0$  or there is an integer  $n_0 \geq 0$  such that  $w_n$  is not periodic for all  $n \geq n_0$ . In the former case all the points  $w_n$  are not critical points of  $f$  since  $w_n \in \mathcal{J}_f$ . In the latter case, there exists an integer  $n_1 \geq 0$  such that  $w_n$  are non-critical points of  $f$  for  $n \geq n_1$ . So in both cases, we have a sequence of points  $\{w_n\}_{n \geq 0}$  in  $T_\infty \setminus \Omega_f$  such that  $f(w_{n+1}) = w_n$ .

Set  $L_n = h^{-1}(w_n)$ . By Theorem A.1 (4),  $L_n$  is a component of  $F^{-1}(L_{n-1})$  and there exists a Jordan domain  $D_0 \supset E_0$  such that  $F^n : D_n \rightarrow D_0$  is a homeomorphism for  $n \geq 1$ , where  $D_n$  is the component of  $F^{-n}(D_0)$  containing  $L_n$ .

Pick an essential Jordan curve in every components of  $\mathfrak{C}$ . They form a Cantor multicurve  $\Gamma_0$ . By Lemma 4.3, there exists an integer  $N \geq 0$  such that for any two distinct points  $z_0, z'_0 \in E_0$ , there is a Jordan arc  $\delta \subset D_0$  connecting  $z_0$  with  $z'_0$ , such that  $\#(\delta \cap \Gamma_0) \leq N$ .

On the other hand, since  $\Gamma_0$  is a Cantor multicurve, there is an integer  $m > 0$  such that for each component  $C$  of  $\mathfrak{C}$ , there are at least  $N + 1$  components of  $F^{-m}(\mathfrak{C})$  contained essentially in  $C$ . Since  $L_m$  crosses a component of  $\mathfrak{C}$ , there exist two distinct points  $z_m, z'_m \in L_m$  such that  $F^{-m}(\Gamma_0)$  has at least  $N + 1$  components separating  $z_m$  from  $z'_m$ .

Now  $F^m(z_m), F^m(z'_m) \in L_0$ . So there exists a Jordan arc  $\delta \subset D_0$  connecting  $F^m(z_m)$  with  $F^m(z'_m)$  such that  $\#(\delta \cap \Gamma_0) \leq N$ . Let  $\delta_m$  be the component of  $F^{-m}(\delta)$  connecting  $z_m$  with  $z'_m$ . Then

$$\#(\delta_m \cap F^{-m}(\Gamma_0)) \leq N$$

since  $F^m : \delta_m \rightarrow \delta$  is a homeomorphism. This contradicts the fact that  $F^{-m}(\Gamma_0)$  has at least  $N + 1$  components separating  $z_m$  from  $z'_m$ .  $\square$

**Corollary 4.4.** *For any  $n \geq 0$  and any distinct components  $E_1, E_2$  of  $\widehat{\mathbb{C}} \setminus F^{-n}(\mathfrak{C})$ ,  $h(E_1)$  is disjoint from  $h(E_2)$ .*

*Proof.*  $E_1$  and  $E_2$  are separated by a component  $A$  of  $F^{-n}(\mathfrak{C})$ . If  $h(E_1) \cap h(E_2) \neq \emptyset$ , pick a point  $w \in h(E_1) \cap h(E_2)$ , then  $h^{-1}(w)$  crosses  $A$ . So  $F^n(h^{-1}(w)) = h^{-1}(f^n(w))$  crosses  $F^n(A)$  by Theorem A.1 (4). This contradicts Lemma 4.2.  $\square$

**Construction of the multi-annulus  $\mathcal{A}$ .** Denote by  $\widehat{E} = h^{-1}(h(E))$  for any subset  $E \subset \widehat{\mathbb{C}}$ . Then  $\widehat{E}$  is also a continuum if  $E$  is a continuum by Theorem A.1 (5).

Denote by  $\mathcal{E} = \widehat{\mathbb{C}} \setminus \mathfrak{C}$ . Then  $F^{-1}(\widehat{\mathcal{E}}) = \widehat{F^{-1}(\mathcal{E})}$  by Theorem A.1 (7). Thus if  $E^1$  is a component of  $F^{-1}(\mathcal{E})$ , then  $\widehat{E^1}$  is a component of  $F^{-1}(\widehat{\mathcal{E}})$  by Corollary 4.4.

For any two disjoint continua  $E_1, E_2 \subset \widehat{\mathbb{C}}$ , we denote by  $A(E_1, E_2)$  the unique annular component of  $\widehat{\mathbb{C}} \setminus (E_1 \cup E_2)$ . For each component  $C$  of  $\mathfrak{C}$ , there are two distinct components  $E_+, E_-$  of  $\mathcal{E}$  such that  $C = A(E_+, E_-)$ . Define  $\widetilde{C} := A(\widehat{E_+}, \widehat{E_-})$ . It is an annulus contained essentially in  $C$  by Corollary 4.4. We claim that the following statements hold:

- (a)  $h^{-1}(h(\widetilde{C})) = \widetilde{C}$ .
- (b)  $\widetilde{C} \cap \widehat{E} = \emptyset$  for any subset  $E \subset \widehat{\mathbb{C}}$  with  $E \cap C = \emptyset$ .
- (c)  $h(\widetilde{C})$  is an annulus homotopic to  $C$  rel  $\mathcal{P}_f$ .

*Proof.* (a) For any point  $z \in \widetilde{C}$ , if  $h^{-1}(h(z))$  is not contained in  $\widetilde{C}$ , then it must intersect with  $E_+ \cup E_-$ . So  $z \in \widehat{E_+} \cup \widehat{E_-}$ . This is a contradiction.

(b) If  $z \in \widetilde{C} \cap \widehat{E}$ , then  $h^{-1}(h(z)) \subset \widetilde{C}$  and hence is disjoint from  $E$ . It contradicts the condition that  $z \in \widehat{E}$ .

(c) Let  $Q_+, Q_-$  be the two components of  $\widehat{\mathbb{C}} \setminus \widetilde{C}$ . Then both  $\widehat{Q_+}$  and  $\widehat{Q_-}$  are disjoint from  $\widetilde{C}$  by (b). Moreover, they are also disjoint from each other since  $h^{-1}(h(z))$  does not cross  $C$  for any point  $z \in \widehat{\mathbb{C}}$  by Lemma 4.2. So  $\widehat{\mathbb{C}} \setminus h(\widetilde{C})$  has exactly two components,  $h(Q_+)$  and  $h(Q_-)$ . Therefore  $h(\widetilde{C})$  is an annulus. Since  $h$  is homotopic to the identity rel  $\mathcal{P}_f$ , the annulus  $h(\widetilde{C})$  is homotopic to  $C$  rel  $\mathcal{P}_f$ .  $\square$

Now let  $\widetilde{\mathfrak{C}}$  be the union of  $\widetilde{C}$  for all the components  $C$  of  $\mathfrak{C}$ . Then  $\widetilde{\mathfrak{C}} \subset \mathfrak{C}$  and it is a multi-annulus homotopic to  $\Gamma$  rel  $\mathcal{P}_f$ . Set  $\mathcal{A}$  to be the union of  $h(\widetilde{C})$  for all the components  $C$  of  $\mathfrak{C}$ . Since  $h(\widetilde{C_1})$  is disjoint from  $h(\widetilde{C_2})$  for distinct components  $C_1, C_2$  of  $\mathfrak{C}$  by (b),  $\mathcal{A}$  is a multi-annulus and homotopic to  $\Gamma$  rel  $\mathcal{P}_f$  by (c). Moreover,  $\mathcal{A}$  is disjoint from a neighborhood of critical cycles of  $f$  since  $h$  is the identity in a neighborhood of critical cycles of  $f$ .

**Construction of  $\mathcal{A}^1$ .** For each component  $C^1$  of  $\mathfrak{C}^1$ , there are two distinct components  $E_+^1, E_-^1$  of  $F^{-1}(\mathcal{E})$  such that  $C^1 = A(E_+^1, E_-^1)$ . Define  $\widetilde{C^1} := A(\widehat{E_+^1}, \widehat{E_-^1})$  as above. It is an annulus essentially contained in  $C^1$ . Moreover, the following statements hold:

- (a1)  $h^{-1}(h(\widetilde{C^1})) = \widetilde{C^1}$ .
- (b1)  $\widetilde{C^1} \cap \widehat{E} = \emptyset$  for any subset  $E \subset \widehat{\mathbb{C}}$  with  $E \cap C^1 = \emptyset$ .
- (c1)  $h(\widetilde{C^1})$  is an annulus homotopic to  $C^1$  rel  $\mathcal{P}_f$ .

Set  $\widetilde{\mathfrak{C}^1}$  to be the union of  $\widetilde{C^1}$  for all the components  $C^1$  of  $\mathfrak{C}^1$ . Set  $\mathcal{A}^1$  to be the union of  $h(\widetilde{C^1})$  for all the components  $C^1$  of  $\mathfrak{C}^1$ . Then it is a multi-annulus essentially contained in  $\mathcal{A}$ .

**Invariance of  $\mathcal{A}$ .** Note that each component of  $\widetilde{\mathfrak{C}}$  is a component of  $\widehat{\mathbb{C}} \setminus \widehat{\mathcal{E}}$  and each component of  $\widetilde{\mathfrak{C}^1}$  is a component of  $\widehat{\mathbb{C}} \setminus F^{-1}(\widehat{\mathcal{E}}) = \widehat{\mathbb{C}} \setminus \widehat{F^{-1}(\mathcal{E})}$ . So  $F : \widetilde{\mathfrak{C}^1} \rightarrow \widetilde{\mathfrak{C}}$  is proper. Since  $\widetilde{\mathfrak{C}} = h^{-1}(\mathcal{A})$  and  $\widetilde{\mathfrak{C}^1} = h^{-1}(\mathcal{A}^1)$ , the map  $f : \mathcal{A}^1 \rightarrow \mathcal{A}$  is also proper.

For any component  $E$  of  $\mathcal{E}$ , there is a unique component  $E^1$  of  $F^{-1}(\mathcal{E})$  such that  $\partial E \subset \partial E^1$ . Moreover,  $E^1 \subset E$  and  $E \setminus E^1$  is a disjoint union of Jordan domains in  $E$ . We claim that  $\widehat{E} \setminus E = \widehat{E^1} \setminus E$ .

Since  $E \supset E^1$ , we have  $\widehat{E} \supset \widehat{E^1}$ . On the other hand, any component  $D$  of  $\widehat{\mathbb{C}} \setminus E$  is a Jordan domain. Assume  $z \in \widehat{E} \cap D$ , then  $h^{-1}(h(z))$  is a full continuum intersecting  $\partial E$  by Theorem A.1 (3). Thus  $h^{-1}(h(z))$  intersects  $\partial E^1$ . Therefore  $z \in \widehat{E^1}$  and hence  $\widehat{E} \setminus E \subset \widehat{E^1} \setminus E$ . The claim is proved.

By the claim,  $\widetilde{\mathfrak{C}^1} \subset \widetilde{\mathfrak{C}}$  and each component of  $\partial \widetilde{\mathfrak{C}}$  is a component of  $\partial \widetilde{\mathfrak{C}^1}$ . Hence  $\mathcal{A}^1 \subset \mathcal{A}$  and each component of  $\partial A$  for any component  $A$  of  $\mathcal{A}$  is a component of  $\partial A^1$  for some component  $A^1$  of  $\mathcal{A}^1$  in  $A$ . So  $f : \mathcal{A}^1 \rightarrow \mathcal{A}$  is an exact annular system.

**Step 4. Uniqueness of  $\mathcal{A}$ .** Suppose that  $f : \mathcal{A}_1^1 \rightarrow \mathcal{A}_1$  is another exact annular system such that  $\mathcal{A}_1$  is homotopic to  $\Gamma$  rel  $\mathcal{P}_f$ . Pick an essential Jordan curve in every components of  $\mathcal{A}$  and  $\mathcal{A}_1$ , respectively. They form two multicurves  $\Gamma_0 \subset \mathcal{A}$  and  $\Gamma_1 \subset \mathcal{A}_1$ . Both of them are homotopic to  $\Gamma$  rel  $\mathcal{P}_f$ . So there exist a neighborhood  $U$  of the critical cycles of  $f$  and a homeomorphism  $\theta_0$  of  $\widehat{\mathbb{C}}$  such that  $\theta_0(\Gamma_0) = \Gamma_1$  and  $\theta_0$  is isotopic to the identity rel  $\mathcal{P}_f \cup U$ . By Theorem A.1, there exist a neighborhood  $V$  of the critical cycles of  $f$  and a sequence  $\{\theta_n\}$  ( $n \geq 1$ ) of homeomorphisms of  $\widehat{\mathbb{C}}$  isotopic to the identity rel  $\mathcal{P}_f \cup V$ , such that  $f \circ \theta_n = \theta_{n-1} \circ f$ . Moreover,  $\{\theta_n\}$  converges uniformly to a continuous map  $h$  of  $\widehat{\mathbb{C}}$ .

It is easy to see that  $h$  is the identity in the Fatou set of  $f$ . On the other hand,  $h$  is also the identity on the Julia set  $\mathcal{J}_f$  since the closure of  $\cup_{n \geq 0} f^{-n}(\mathcal{P}_f)$  contains  $\mathcal{J}_f$  and  $\theta_n$  is the identity on  $f^{-n}(\mathcal{P}_f)$ . So  $\{\theta_n\}$  converges uniformly to the identity.

For each component  $A$  of  $\mathcal{A}$ , set  $A(n, \Gamma_0)$  to be the closed annulus bounded by two curves in  $A \cap (f|_{\mathcal{A}_1})^{-n}(\Gamma_0)$  such that  $A \cap (f|_{\mathcal{A}_1})^{-n}(\Gamma_0) \subset A(n, \Gamma_0)$ . Then  $\theta_n(A(n, \Gamma_0)) \subset \mathcal{A}_1$  since  $\theta_n(f^{-n}(\Gamma_0)) = f^{-n}(\Gamma_1)$ . By Proposition 3.2, for any compact set  $G \subset A$ ,  $G \subset A(n, \Gamma_0)$  as  $n$  is large enough. So  $A \subset \mathcal{A}_1$  since  $\{\theta_n\}$  converges uniformly to the identity. It follows that  $\mathcal{A} \subset \mathcal{A}_1$ . By symmetry, we have  $\mathcal{A} = \mathcal{A}_1$ .

**Step 5. Properties of  $\mathcal{J}_g$ .** We want to prove that  $\mathcal{J}_g \subset \mathcal{J}_f$ . Assume by contradiction that there is a point  $z \in \mathcal{J}_g \setminus \mathcal{J}_f$ . Then  $\{f^n(z)\}_{n \geq 0}$  converges to a super-attracting cycle of  $f$  as  $n \rightarrow \infty$ . But  $f^n(z) \in g^n(\mathcal{J}_g) \subset \mathcal{J}_g$ . Thus  $\overline{\mathcal{A}}$  contains a critical cycle. This is a contradiction since  $\mathcal{A}$  is disjoint from a neighborhood of critical cycles.

There is a singular conformal metric  $\rho$  on  $\widehat{\mathbb{C}}$  where the singularities may occur at  $\mathcal{P}_f$  such that  $f$  is strictly expanding on  $(\widehat{\mathbb{C}}, \rho)$  except in a neighborhood of super-attracting cycles (e.g., the hyperbolic metric on the orbifold of  $f$ , refer to [8, 25, 26]). Applying Theorem 3.9, we see that every component of  $\mathcal{J}_g$  is a Jordan curve. In particular, there

exists a wandering Jordan curve on  $\mathcal{J}_g$  by Corollary 3.6.

## 5 Decomposition and renormaliation

We will prove Theorem 1.3 in this section. At first, we want to introduce the definition of rational-like maps and prove a straightening theorem.

### 5.1 Rational-like maps

**Definition 3.** Let  $U \Subset V$  be two finitely-connected domains in  $\widehat{\mathbb{C}}$ . A map  $g : U \rightarrow V$  is called a **rational-like map** if

- (1)  $g$  is holomorphic and proper with  $\deg g \geq 2$ ,
- (2) the orbit of every critical point of  $g$  (if any) stays in  $U$ , and
- (3) each component of  $\widehat{\mathbb{C}} \setminus U$  contains at most one component of  $\widehat{\mathbb{C}} \setminus V$ .

The **filled-in Julia set** of  $g$  is defined by

$$\mathcal{K}_g = \bigcap_{n \geq 0} g^{-n}(V).$$

We say that a rational-like map  $g : U \rightarrow V$  is a **renormalization** of a rational map  $f$  if  $g = f^p|_U$  for some integer  $p \geq 1$  and  $\deg g < \deg f^p$ .

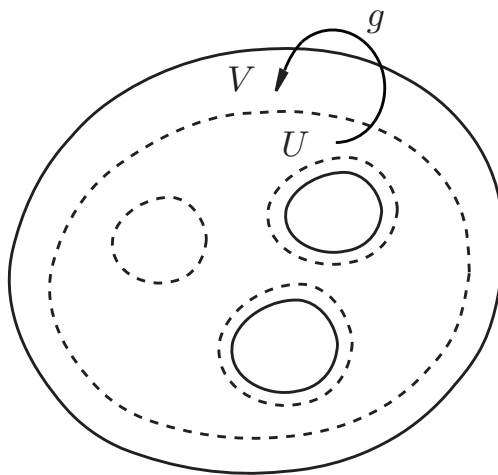


Figure 1. A rational-like map.

**Remark.** A rational-like map here is actually a repelling system of constant complexity in [6].

**Proposition 5.1.** Let  $g : U_1 \rightarrow U_0$  be a rational-like map. Then  $g^{-n}(U_0)$  is connected for any  $n \geq 1$ . The filled-in Julia set  $\mathcal{K}_g$  is a continuum.

*Proof.* Pick a domain  $V_0 \Subset U_0$  such that every component of  $\widehat{\mathbb{C}} \setminus V_0$  contains exactly one component of  $\widehat{\mathbb{C}} \setminus U_0$ ,  $U_1 \Subset V_0$  and every component of  $\partial V_0$  is a Jordan curve. Set  $V_1 := g^{-1}(V_0)$ . Then  $V_1 \Subset V_0$ , every component of  $\widehat{\mathbb{C}} \setminus V_1$  contains at most one component of  $\widehat{\mathbb{C}} \setminus V_0$  and each component of  $\partial V_1$  is a Jordan curve.

Since every component of  $\widehat{\mathbb{C}} \setminus V_1$  contains at most one component of  $\widehat{\mathbb{C}} \setminus V_0$ , each component  $W$  of  $V_0 \setminus \overline{V_1}$  is either a disk or an annulus. In the latter case, one of the component of the boundary  $\partial W$  is a component of the boundary  $\partial V_0$  and the other is a component of the boundary  $\partial V_1$ .

Denote by  $V_n = g^{-n}(V_0)$  for  $n > 1$ . Then  $V_{n+1} \Subset V_n$  for  $n \geq 1$ . Since all the critical orbits of  $g$  stay in  $U_1$  and thus in  $\mathcal{K}_g$ , each component  $W$  of  $V_1 \setminus \overline{V_2}$  is also either a disk or an annulus. In the latter case, one of the component of  $\partial W$  is a component of  $\partial V_1$  and the other is a component of  $\partial V_2$ . Therefore,  $V_2$  is also connected and every component of  $\widehat{\mathbb{C}} \setminus V_2$  contains at most one component of  $\widehat{\mathbb{C}} \setminus V_1$ . Inductively, we have that  $V_{n+1}$  is connected and every component of  $\widehat{\mathbb{C}} \setminus V_{n+1}$  contains at most one component of  $\widehat{\mathbb{C}} \setminus V_n$ . It follows that  $\mathcal{K}_g$  is a connected compact set.  $\square$

Similar to Douady-Hubbard's polynomial-like map theory [9], we may have a straightening theorem for rational-like maps with a slightly different proof.

**Theorem 5.2.** *Let  $g : U \rightarrow V$  be a rational-like map, then there is a rational map  $f$  with  $\deg f = \deg g$  and a quasiconformal map  $\phi$  of  $\widehat{\mathbb{C}}$  such that:*

- (a)  $f \circ \phi = \phi \circ g$  in a neighborhood of  $\mathcal{K}_g$ ,
- (b) the complex dilatation  $\mu_\phi$  of  $\phi$  satisfying  $\mu_\phi(z) = 0$  for a.e.  $z \in \mathcal{K}_g$ ,
- (c)  $\mathcal{J}_f = \partial\phi(\mathcal{K}_g)$ , and
- (d) each component of  $\widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g)$  contains at most one point of  $\mathcal{P}_f$ .

*Moreover, the rational map  $f$  is unique up to holomorphic conjugation.*

*Proof.* Pick a domain  $V_1 \Subset V$  such that every component of  $\widehat{\mathbb{C}} \setminus V_1$  contains exactly one component of  $\widehat{\mathbb{C}} \setminus V$ ,  $U \Subset V_1$  and every component of  $\partial V_1$  is a quasicircle. Then  $U_1 := g^{-1}(V_1) \Subset V_1$ , every component of  $\widehat{\mathbb{C}} \setminus U_1$  contains at most one component of  $\widehat{\mathbb{C}} \setminus V_1$  and each component of  $\partial U_1$  is a quasicircle.

Let  $E_1, \dots, E_m$  be the components of  $\mathcal{E} := \widehat{\mathbb{C}} \setminus \overline{V_1}$ . Let  $B_1, \dots, B_n$  be the components of  $\mathcal{B} := \widehat{\mathbb{C}} \setminus \overline{U_1}$  such that  $B_i \supset E_i$  for  $1 \leq i \leq m$ . Then  $\mathcal{E} \Subset \mathcal{B}$ . Define a map  $\tau$  on the index set by  $\tau(i) = j$  if  $g(\partial B_i) = \partial E_j$ .

Let  $D_i \subset \mathbb{C}$  ( $i = 1, \dots, n$ ) be round disks centered at  $a_i$  with unit radius such that their closures are pairwise disjoint. Denote their union by  $\mathcal{D}$ . Define a map  $Q$  on  $\mathcal{D}$  by

$$Q(z) = r(z - a_i)^{d_i} + a_{\tau(i)}, \quad z \in D_i,$$

where  $0 < r < 1$  is a constant and  $d_i = \deg(g|_{\partial E_i})$ . Then  $Q(D_i) \Subset D_{\tau(i)}$ . Denote by  $D_{\tau(i)}(r) := Q(D_i)$  and  $\mathcal{D}(r) = Q(\mathcal{D})$ .

Let  $\psi : \mathcal{E} \rightarrow \mathcal{D}(r)$  be a conformal map such that  $\psi(E_i) = D_i(r)$ . It can be extended to a quasiconformal map in a neighborhood of  $\mathcal{E}$  since the components of  $\mathcal{E}$  are quasidisks with pairwise disjoint closures. Since  $Q : \partial\mathcal{D} \rightarrow \partial\mathcal{D}(r)$  and  $g : \partial\mathcal{B} \rightarrow \partial\mathcal{E}$  are coverings with

same degrees on corresponding components, there is a homeomorphism  $\psi_1 : \partial\mathcal{B} \rightarrow \partial\mathcal{D}$  such that  $\psi \circ g = Q \circ \psi_1$ .

Since each component of  $\partial\mathcal{B}$  is a quasicycle, the conformal map  $\psi : \mathcal{E} \rightarrow \mathcal{D}(r)$  can be extended to a homeomorphism  $\psi : \overline{\mathcal{B}} \rightarrow \overline{\mathcal{D}}$  such that  $\psi|_{\partial\mathcal{B}} = \psi_1$  and  $\psi$  is quasiconformal on  $\mathcal{B}$ . Define a map

$$G = \begin{cases} g & \text{on } U_1, \\ \psi^{-1} \circ Q \circ \psi & \text{on } \overline{\mathcal{B}}. \end{cases}$$

Then  $G$  is a quasiregular branched covering of  $\widehat{\mathbb{C}}$ . Set  $\mathcal{O} := \psi^{-1}(\{a_1, \dots, a_n\})$ . Then

$$G(\mathcal{O}) \subset \mathcal{O} \quad \text{and} \quad \mathcal{P}_G \setminus \mathcal{K}_g \subset \mathcal{O}$$

since no critical point of  $g$  escapes. Moreover, for each point  $z \in \widehat{\mathbb{C}} \setminus \mathcal{K}_g$ , its forward orbit  $\{G^n(z)\}$  converges to the invariant set  $\mathcal{O}$ .

By the measurable Riemann mapping theorem, there is a quasiconformal map  $\Phi$  of  $\widehat{\mathbb{C}}$  such that its complex dilatation satisfies  $\mu_\Phi = 0$  on  $U_1$  and  $\mu_\Phi = \mu_\psi$  on  $\mathcal{B}$ . Set  $F := \Phi \circ G \circ \Phi^{-1}$ . Then  $F$  is holomorphic in the interior of  $\Phi(g^{-1}(U_1) \cup \mathcal{E})$ .

For any orbit  $\{F^n(z)\}_{n \geq 0}$ , if  $z$  is not contained in the interior of  $\Phi(g^{-1}(U_1) \cup \mathcal{E})$ , then  $z$  is contained in either  $\Phi(\overline{U_1}) \setminus \Phi \circ g^{-1}(U_1)$  or the closure of  $\Phi(\mathcal{B}) \setminus \Phi(\mathcal{E})$ . In the latter case,  $F(z) \in \Phi(\mathcal{E})$  and thus  $F^2(z)$  is contained in the interior of  $\Phi(\mathcal{E})$ . In the former case,  $F(z) \in \Phi(\mathcal{B}) \setminus \Phi(\mathcal{E})$  and thus  $F^3(z)$  is contained in the interior of  $\Phi(\mathcal{E})$ . Thus  $F^n(z)$  is contained in the interior of  $\Phi(\mathcal{E})$  for  $n \geq 3$  in both cases. This shows that every orbit of  $F$  passes through the closure of  $\widehat{\mathbb{C}} \setminus \Phi(g^{-1}(U_1) \cup \mathcal{E})$  at most three times. Applying Shishikura's Surgery Principle (see Lemma 15 in [1]), there is quasiconformal map  $\Phi_1 : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  such that  $f = \Phi_1 \circ F \circ \Phi_1^{-1}$  is a rational map. Moreover,  $\mu_{\Phi_1}(z) = 0$  for *a.e.*  $z \in \Phi(\mathcal{K}_g)$ . Set  $\phi = \Phi_1 \circ \Phi$ . Then  $f \circ \phi = \phi \circ g$  on  $U_1$  and  $\mu_\phi(z) = 0$  for *a.e.*  $z \in \mathcal{K}_g$ .

For a compact set  $E \subset \widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g)$ , its forward orbit  $\{f^n(E)\}$  converges to the invariant set  $\phi(\mathcal{O}) \subset \mathcal{F}_f$ . Moreover,  $\mathcal{P}_f \setminus \phi(\mathcal{K}_g) \subset \phi(\mathcal{O})$ . So  $\widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g) \subset \mathcal{F}_f$ . Since  $\phi(\mathcal{K}_g)$  is completely invariant under  $f$ , we have  $\partial\phi(\mathcal{K}_g) = \mathcal{J}_f$ .

If there is another rational map  $f_1$  satisfying the conditions of the theorem, then there is a quasiconformal map  $\theta$  of  $\widehat{\mathbb{C}}$  such that  $f_1 \circ \theta = \theta \circ f$  in a neighborhood of  $\phi(\mathcal{K}_g)$  and  $\mu_\theta(z) = 0$  for *a.e.*  $z \in \phi(\mathcal{K}_g)$ .

Let  $W$  be a periodic Fatou domain of  $f$  in  $\widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g)$  with period  $p \geq 1$ . Then  $W$  is simply-connected and contains exactly one point  $z_0 \in \mathcal{P}_f$ , which is a super-attracting periodic point. Therefore there is a conformal map  $\eta$  from  $W$  onto the unit disc  $\mathbb{D}$  such that  $\eta(z_0) = 0$  and  $\eta \circ f^p \circ \eta^{-1}(z) = z^d$  with  $d = \deg_{z_0} f^p > 1$ . On the other hand, let  $z_1 \in \theta(W)$  be the super-attracting periodic point of  $f_1$ , then there is a conformal map  $\eta_1 : \theta(W) \rightarrow \mathbb{D}$  such that  $\eta_1(z_1) = 0$  and  $\eta_1 \circ f_1^p \circ \eta_1^{-1}(z) = z^d$ . Therefore

$$\eta_1 \circ \theta \circ f^p \circ \theta^{-1} \circ \eta_1^{-1}(z) = z^d$$

in a neighborhood of  $\partial\mathbb{D}$  in  $\mathbb{D}$ . This shows that  $T = \eta_1 \circ \theta \circ \eta^{-1}$  is a rotation on  $\partial\mathbb{D}$  (see



the commutative diagram below).

$$\begin{array}{ccccccc}
 \mathbb{D} & \xleftarrow{\eta} & W & \xrightarrow{\theta} & \theta(W) & \xrightarrow{\eta_1} & \mathbb{D} \\
 \downarrow z \mapsto z^d & & \downarrow f^p & & \downarrow f_1^p & & \downarrow z \mapsto z^d \\
 \mathbb{D} & \xleftarrow{\eta} & W & \xrightarrow{\theta} & \theta(W) & \xrightarrow{\eta_1} & \mathbb{D}
 \end{array}$$

Let  $\theta_W = \eta_1^{-1} \circ T \circ \eta$ . Then  $\theta_W : W \rightarrow \theta(W)$  is holomorphic,  $\theta_W = \theta$  on the boundary  $\partial W$  and  $f_1 \circ \theta_W = \theta_W \circ f$ .

Define  $\Theta_0 : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  by  $\Theta_0 = \theta_W$  on all the super-attracting Fatou domains of  $f$  in  $\widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g)$ , and  $\Theta_0 = \theta$  otherwise. Then  $\Theta_0$  is a quasiconformal map and  $\Theta_0 \circ f = f_1 \circ \Theta_0$  on the union of  $\phi(\mathcal{K}_g)$  and all the super-attracting Fatou domains of  $f$  in  $\widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g)$ . Pullback  $\Theta_0$ , we have a sequence of quasiconformal maps  $\Theta_n : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  such that  $\Theta_0 \circ f^n = f_1^n \circ \Theta_n$ , in particular, the following diagram commutes.

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow f & & \downarrow f_1 \\
 \widehat{\mathbb{C}} & \xrightarrow{\Theta_2} & \widehat{\mathbb{C}} \\
 \downarrow f & & \downarrow f_1 \\
 \widehat{\mathbb{C}} & \xrightarrow{\Theta_1} & \widehat{\mathbb{C}} \\
 \downarrow f & & \downarrow f_1 \\
 \widehat{\mathbb{C}} & \xrightarrow{\Theta_0} & \widehat{\mathbb{C}}_0
 \end{array}$$

It is easy to check that  $\Theta_n$  uniformly converges to a holomorphic conjugacy from  $f$  to  $f_1$ .  $\square$

## 5.2 Renormalization

Let  $f$  be a post-critically rational map with a stable Cantor multicurve  $\Gamma$ . By Theorem 4.1, there is a unique multi-annulus  $\mathcal{A} \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$  homotopic rel  $\mathcal{P}_f$  to  $\Gamma$  such that  $g = f|_{\mathcal{A}^1} : \mathcal{A}^1 \rightarrow \mathcal{A}$  is an exact annular system, where  $\mathcal{A}^1$  is the union of the components of  $f^{-1}(\mathcal{A})$  homotopic rel  $\mathcal{P}_f$  to curves in  $\Gamma$ . Moreover,  $\mathcal{J}_g \subset \mathcal{J}_f$  and each component of  $\mathcal{J}_g$  is a Jordan curve. Denote by

$$\mathcal{J}(\Gamma) = \bigcup_{n>0} f^{-n}(\mathcal{J}_g) \quad \text{and} \quad \mathcal{K}(\Gamma) = \widehat{\mathbb{C}} \setminus \mathcal{J}(\Gamma).$$

Then both of them are completely invariant. Note that

$$\partial f^{-n+1}(\mathcal{A}) \subset \partial f^{-n}(\mathcal{A}) \subset \mathcal{K}(\Gamma)$$

for  $n \geq 1$ . Consequently, each component of  $\mathcal{J}(\Gamma)$  is also a Jordan curve.

Recall that a connected subset  $E \subset \widehat{\mathbb{C}}$  is of simple type (w.r.t.  $\mathcal{P}_f$ ) if there exists either a simply-connected domain  $U \subset \widehat{\mathbb{C}}$  such that  $E \subset U$  and  $U$  contains at most one

point in  $\mathcal{P}_f$ , or an annulus  $A \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$  such that  $E \subset A$ ; and is of complex type (w.r.t.  $\mathcal{P}_f$ ) otherwise. Since  $f(\mathcal{P}_f) \subset \mathcal{P}_f$ , for each simple type continuum  $E \subset \widehat{\mathbb{C}}$ , each component of  $f^{-1}(E)$  is also simple type.

Denote by  $\mathcal{K}_c$  the union of complex type components of  $\mathcal{K}(\Gamma)$ .

**Proposition 5.3.**  *$f(\mathcal{K}_c) = \mathcal{K}_c$  and  $\mathcal{K}_c$  is compact which has exactly  $\#\Gamma + 1$  components.*

*Proof.* Denote by  $N = \#\Gamma$ . Then  $\widehat{\mathbb{C}} \setminus \mathcal{A}$  has exactly  $N + 1$  components  $B_0^0, \dots, B_N^0$  and each  $B_i^0$  is a complex type continuum since the essential Jordan curves in  $\mathcal{A}$  are essential in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ .

Let  $A_n, A_m$  be a component of  $f^{-n}(\mathcal{A})$  and  $f^{-m}(\mathcal{A})$ , respectively for  $n \geq m \geq 0$ . Then either they are disjoint or one contains another. For each  $B_i^0$ , let  $A$  be a component of  $f^{-n}(\mathcal{A})$  for some  $n \geq 1$ . Then either  $A \cap B_i^0 = \emptyset$  or  $A \subset B_i^0$ . In the latter case, the essential Jordan curve in  $A$  is either null-homotopic or peripheral since  $\Gamma$  is stable. Thus for each  $n \geq 1$ ,  $B_i^0 \setminus f^{-n}(\mathcal{A})$  has exactly one complex type component, denote it by  $B_i^n$ . Moreover  $B_i^{n+1} \subset B_i^n$  and  $f(B_i^{n+1}) = B_j^n$  for some  $j = 1, \dots, N$  since  $\partial \mathcal{A}_n \subset \partial \mathcal{A}_{n+1}$ . Denote by

$$K_i = \bigcap_{n \geq 0} B_i^n \text{ for } i = 0, \dots, N.$$

Then  $f(K_i) = K_j$  for some  $0 \leq j \leq N$  and each  $K_i$  is a complex type component of  $\mathcal{K}(\Gamma)$ .

Each component of  $\widehat{\mathbb{C}} \setminus \bigcup_{i=0}^N K_i$  is either a component of  $\mathcal{A}$  or a simply-connected domain contains at most one point of  $\mathcal{P}_f$ . Therefore each component of  $\mathcal{K}(\Gamma) \setminus \bigcup_{i=0}^N K_i$  is of simple type. Thus  $\mathcal{K}_c = \bigcup_{i=0}^N K_i$ .  $\square$

**Theorem 5.4.** *Let  $K$  be a periodic component of  $\mathcal{K}_c$  with period  $p \geq 1$ . Then there exist domains  $U \Subset V$  in  $\widehat{\mathbb{C}}$  such that  $K \subset U$  and  $g = f^p|_U : U \rightarrow V$  is a renormalization of  $f$  with filled-in Julia set  $\mathcal{K}_g = K$ .*

*Proof.* Let  $B_0, \dots, B_{p-1}$  be the components of  $\widehat{\mathbb{C}} \setminus \mathcal{A}$  such that  $K \subset B_0$  and  $f^i(K) \subset B_i$  for  $0 < i < p$ . Let  $A_1, \dots, A_n$  be the components of  $\mathcal{A}$  whose boundary intersects with  $B_0$ . Set

$$W' = B_0 \cup \bigcup_{i=1}^n A_i.$$

It is a finitely-connected domain. Let  $W'_1$  be the component of  $f^{-p}(W')$  containing  $K$ . Then  $W'_1 \subset W'$  and each component of  $W' \setminus \overline{W'_1}$  is either an annulus disjoint from  $\mathcal{P}_f$  or a disk containing at most one point of  $\mathcal{P}_f$  since  $\Gamma$  is stable.

Each  $A_i$  contains exactly one component of  $W'_1 \setminus K$ , denoted by  $A_i^p$ , which is a component of  $f^{-p}(\mathcal{A})$  and shares a common boundary component with  $A_i$ . By relabelling the index of  $A_i$ , we may assume that

$$f^p(A_1^p) = A_2, \dots, f^p(A_{q-1}^p) = A_q \text{ and } f^p(A_q^p) = A_1, q \geq 1.$$

Then there is at least one of them, say  $A_i$ , such that  $A_i^p \subsetneq A_i$ . Otherwise  $A_i^p = A_i$  for  $1 \leq i \leq q$  and hence  $f^{qp}(A_1) = A_1$ . It contradicts the fact that  $f : \mathcal{A}^1 \rightarrow \mathcal{A}$  is an annular system.

Assume that  $A_1^p \subsetneq A_1$ . Then there exists a Jordan curve  $\gamma_1$  essentially contained in  $A_1$  such that it is disjoint from  $A_1^p$ . Let  $\gamma'_q$  be the component of  $f^{-p}(\gamma_1)$  in  $A_q^p$ . Then we can find a Jordan curve  $\gamma_q$  essentially contained in  $A_q$  such that  $\gamma'_q$  separates  $\gamma_q$  from  $K$ . Inductively, for  $2 \leq i < q$ , let  $\gamma'_i$  be the component of  $f^{-p}(\gamma_{i+1})$  in  $A_i^p$ , we can find a Jordan curve  $\gamma_i$  essentially contained in  $A_i$  such that  $\gamma'_i$  separates  $\gamma_i$  from  $K$ . Since  $\gamma_1$  is disjoint from  $A_1^p$ , the component of  $f^{-p}(\gamma_2)$  in  $A_1$  separates  $\gamma_1$  from  $K$  as well.

Do this process for each cycle, we have a Jordan curve  $\gamma_i$  essentially contained in each periodic annulus  $A_i$  such that if  $f^p(A_i^p) = A_j$ , then the component of  $f^{-p}(\gamma_j)$  in  $A_i$  separates  $\gamma_i$  from  $K$ . If  $A_i$  is not periodic but  $A_j = f^p(A_i^p)$  is periodic, then there is always a Jordan curve  $\gamma_i$  essentially contained in  $A_i$  such that the component of  $f^{-p}(\gamma_j)$  in  $A_i$  separates  $\gamma_i$  from  $K$ .

In summary, we have a Jordan curve  $\gamma_i \subset A_i$  for each  $A_i$  such that if  $f^p(A_i^p) = A_j$ , then the component of  $f^{-p}(\gamma_j)$  in  $A_i^p$  separates  $\gamma_i$  from  $K$ . Let  $W \subset W'$  be the domain bounded by the curves  $\gamma_i$  defined above. Then  $W_1 \Subset W$ , where  $W_1$  is the component of  $f^{-p}(W)$  containing  $K$ , and each component of  $W \setminus \overline{W_1}$  is either an annulus disjoint from  $\mathcal{P}_f$  or a disk containing at most one point of  $\mathcal{P}_f$ .

Let  $W_n$  be the component of  $f^{-np}(W)$  containing  $K$  for  $n \geq 2$ . Then  $W_n \Subset W_{n-1}$  and each component of  $W \setminus \overline{W_1}$  is either an annulus disjoint from  $\mathcal{P}_f$ , or a disk which contains at most one point of  $\mathcal{P}_f$ .

Since  $\mathcal{P}_f$  is finite, there is an integer  $N \geq 1$  such that  $W_n \cap \mathcal{P}_f = W_N \cap \mathcal{P}_f$  for  $n \geq N$ . Set  $U = W_{N+1}$ ,  $V = W_N$  and  $g := f^p|_U : U \rightarrow V$ . Then every critical points of  $g$  stay in  $U$ . By Proposition 3.1, there is an integer  $n \geq 1$  such that  $\deg(f^n|_A) \geq 2$  for all the components  $A$  of  $\mathcal{A}^n$ . So we have  $\deg g \geq 2$ . Therefore  $g : U \rightarrow V$  is a rational-like map.

Now we want to show that  $\deg g < \deg f^p$ . Otherwise  $\mathcal{J}_f \subset \mathcal{K}_g$ . But we know that the Julia set of the annular system  $f : \mathcal{A}^1 \rightarrow \mathcal{A}$  is contained in  $\mathcal{J}_f$ . This is impossible. So  $\deg g < \deg f^p$ . It follows that  $g$  is a renormalization of  $f$ .  $\square$

From Theorem 5.2, Theorem 5.4 and Theorem 2.1 in [25], we have:

**Corollary 5.5.** *Let  $K$  be a component of  $\mathcal{K}_c$ . For each component  $W$  of  $\widehat{\mathbb{C}} \setminus K$ , its boundary  $\partial W$  is locally connected.*

### 5.3 Topology of $\mathcal{K}(\Gamma)$

Recall that  $\mathcal{K}_c$  is the union of complex type components of  $\mathcal{K}(\Gamma)$ . Since  $\#\mathcal{P}_f$  is finite, there exists an integer  $l \geq 1$  such that each component of  $f^{-(l+1)}(\mathcal{K}_c) \setminus f^{-l}(\mathcal{K}_c)$  is disjoint from  $\mathcal{P}_f$ , and is either null-homotopic or homotopic to a component of  $f^{-1}(\mathcal{K}_c)$  rel  $\mathcal{P}_f$ . Denote by  $\mathcal{K}_1$  the union of components of  $f^{-l}(\mathcal{K}_c)$  which are not null-homotopic. Then  $f(\mathcal{K}_1) \subset \mathcal{K}_1$ .

Denote by  $\mathcal{U} = \widehat{\mathbb{C}} \setminus \mathcal{K}_1$ . Then  $f^{-1}(\mathcal{U}) \subset \mathcal{U}$ . Each component of  $\mathcal{U}$  is either a simply-connected domain contains at most one point in  $\mathcal{P}_f$ , or a component of  $f^{-l}(\mathcal{A})$ . It can be decomposed into:

$$\mathcal{U} = \mathcal{R} \sqcup \mathcal{G} \sqcup \mathcal{D} \sqcup \mathcal{Q}$$

where  $\mathcal{R}$  is the union of annular components of  $\mathcal{U}$ , which are components of  $f^{-l}(\mathcal{A})$  that are not null-homotopic,  $\mathcal{G}$  is the union of simply-connected components of  $\mathcal{U}$  which intersects with  $\mathcal{P}_f$ ,  $\mathcal{D}$  is the union of components  $D$  of  $\mathcal{U} \setminus (\mathcal{R} \cup \mathcal{G})$  such that either  $D \cap f^{-1}(\mathcal{P}_f) \neq \emptyset$  or  $D \cap f^{-1}(\mathcal{R}) \neq \emptyset$ , and  $\mathcal{Q}$  is the union of simply-connected components of  $\mathcal{U}$  which are disjoint from  $f^{-1}(\mathcal{P}_f \cup \mathcal{R})$ .

Obviously, both  $\mathcal{R}$  and  $\mathcal{G}$  have only finitely many components. Since  $f^{-1}(\mathcal{U}) \subset \mathcal{U}$ , we know that  $\mathcal{D}$  has only finitely many components.

**Lemma 5.6.** *For each component  $G$  of  $\mathcal{G}$ ,  $f^{-1}(G) \cap \mathcal{R} = \emptyset$  and  $G$  contains exactly one component of  $f^{-1}(\mathcal{G})$ .*

*Proof.* Assume that  $f^{-1}(G)$  has a component  $W \subset \mathcal{R}$ , then  $G = f(W) \subset f^{-l+1}(\mathcal{A})$ . But  $f^{-l+1}(\mathcal{A})$  is disjoint from  $\mathcal{P}_f$ . Contradiction.

Denote by  $a \in G$  be the unique point of  $\mathcal{P}_f$ . Then  $f(a)$  is also contained in a component  $G_1$  of  $\mathcal{G}$ . Thus  $f^{-1}(G_1)$  has a component in  $G$ .

Let  $W_1$  be the component of  $\widehat{\mathbb{C}} \setminus \mathcal{K}_c$  that contains  $f(a)$ . Let  $W_0$  be the component of  $f^{-1}(W_1)$  that contains the point  $a$ . Then  $G \subset W_0$ . Since  $W_0$  contains a unique point in  $f^{-1}(\mathcal{P}_f)$ , so does  $G$ . Therefore  $G$  contains exactly one component of  $f^{-1}(\mathcal{G})$ .  $\square$

Denote by  $\mathcal{U}_n = f^{-n}(\mathcal{U})$  for  $n \geq 0$ . Then  $\mathcal{U}_{n+1} \subset \mathcal{U}_n$ . Each component of  $\mathcal{U}_n$  is either a simply-connected domain contains at most one point in  $\mathcal{P}_f$ , or a component of  $f^{-n-l}(\mathcal{A})$ .

Denote by  $\mathcal{K}_s = \mathcal{K}(\Gamma) \setminus \bigcup_{n \geq 0} f^{-n}(\mathcal{K}_c)$ . For each component  $K$  of  $\mathcal{K}_s$ , denote by  $U_n(K)$  the component of  $\mathcal{U}_n$  that contains  $K$  for each  $n \geq 0$ .

**Lemma 5.7.** *For each component  $K$  of  $\mathcal{K}_s$  and each  $n \geq 0$ , there exists an integer  $m > n$  such that  $U_m(K) \Subset U_n(K)$ .*

*Proof.* If  $U_n(K)$  is an annulus, then it is a component of  $f^{-n-l}(\mathcal{A})$ . Pick a point  $z \in K$ . From Proposition 3.3 (2), there exist two components  $A_1, A_2$  of  $f^{-m-l}(\mathcal{A})$  for some  $m > n$  such that both  $A_1$  and  $A_2$  are contained in  $U_n(K)$  essentially and the 2-connected continuum between  $A_1$  and  $A_2$ , denoted by  $E$ , contains the point  $z$ . Note that both  $\partial A_1$  and  $\partial A_2$  are contained in  $f^{-m-l}(\mathcal{K}_c)$ . Thus  $U_m(K) \subset E$  and hence  $U_m(K) \Subset U_n(K)$ .

The same argument works when  $U_n(K)$  is simply-connected.  $\square$

**Lemma 5.8.** *Let  $K$  be a component of  $\mathcal{K}_s$  and  $n \geq 0$  be an integer. If  $U_n(K)$  is an annulus, then there exists an integer  $m > n$  such that  $U_m(K)$  is either simply-connected or an annulus but is not contained in  $U_n(K)$  essentially.*

*Proof.* Otherwise,  $\{U_m(K)\}$  are all annuli and  $U_{m+1}(K)$  is contained in  $U_m(K)$  essentially for all  $m \geq n$ . Then  $\bigcap_{n \geq 0} U_n(K)$  is a component of  $\mathcal{J}(\Gamma)$ . Contradiction.  $\square$

Let  $K$  be a component of  $\mathcal{K}_s$ . By Lemma 5.8, we know that as  $n$  is large enough, either  $U_n(K)$  is simply-connected, or  $U_n(K)$  is an annulus and one component  $E_n(K)$  of  $\widehat{\mathbb{C}} \setminus U_n(K)$  contains at most one point of  $\mathcal{P}_f$ . Denote by  $V_n(K) = U_n(K) \cup E_n(K)$  in the case or  $V_n(K) = U_n(K)$  otherwise. Then  $V_n(K)$  is a simply-connected domain contains at

most one point of  $\mathcal{P}_f$  as  $n$  is large enough and  $V_{n+1}(K) \subset V_n(K)$  for all  $n \geq 0$ . Moreover, for each integer  $n \geq 0$ , there exists an integer  $m > n$  such that  $V_m(K) \subset U_n(K)$ . Thus  $\cap_{n>0} V_n(K) = \cap_{n>0} U_n(K)$  and it is disjoint from  $\cup_{n \geq 0} f^{-n}(\mathcal{K}_c)$  and  $\mathcal{J}(\Gamma)$ . Thus we have

$$K = \bigcap_{n>0} U_n(K) = \bigcap_{n>0} V_n(K).$$

**Corollary 5.9.** *Each component of  $\mathcal{K}(\Gamma)$  is compact.*

**Proposition 5.10.** *Let  $K$  be a periodic component of  $\mathcal{K}_s$ . Then  $K$  is either a single point or the closure of a quasi-disk, which is a periodic Fatou domain of  $f$ .*

*Proof.* Let  $K$  be a periodic component of  $\mathcal{K}_s$  with period  $p \geq 1$ . Then as  $n$  is large enough,  $V_n(K)$  is simply-connected and  $f^p : V_{n+p}(K) \rightarrow V_n(K)$  is proper with at most one critical point. Moreover  $V_{n+p}(K) \Subset V_n(K)$  by Lemma 5.7. If  $K$  contains no super-attracting periodic points of  $f$ , then

$$\deg(f^p : V_{n+p}(K) \rightarrow V_n(K)) = 1$$

as  $n$  is large enough. Thus  $K$  is a single point. Otherwise  $f^p : V_{n+p}(K) \rightarrow V_n(K)$  is a polynomial-like map and  $K$  is the closure of a quasi-disk, which is a periodic Fatou domain of  $f$ .  $\square$

**Lemma 5.11.** *Let  $K$  be a wandering component of  $\mathcal{K}(\Gamma)$ . Then there exists an integer  $n > 0$  such that  $f^n(K) \subset \mathcal{D}$ .*

*Proof.* At first, we claim that there exists an integer  $m > 0$  such that  $f^m(K) \subset \mathcal{D} \cup \mathcal{Q}$ . Assume by contradiction that  $f^n(K) \subset \mathcal{R} \cup \mathcal{G}$  for all  $n > 0$ .

Suppose that there exists an integer  $n_0 > 0$  such that  $f^n(K) \subset \mathcal{R}$  for all  $n \geq n_0$ . Then there is a component  $A$  of  $\mathcal{A}$  such that  $f^{n_0+l}(K) \subset A$  since each component of  $\mathcal{R}$  is a component of  $f^{-l}(\mathcal{A})$  which is not null-homotopic. On the other hand, there are at least one component of  $\mathcal{K}_1$  contained in  $A$ . Thus each component of  $A \setminus \mathcal{K}_1$  is either a component of  $\mathcal{D} \cup \mathcal{Q}$  or is contained in  $\mathcal{A}^1$ . Thus  $f^{n_0+l}(K) \subset \mathcal{A}^1$  by the assumption. Therefore  $f^{n+l}(K) \subset \mathcal{A}^1$  for all  $n \geq n_0$ . Consequently,  $K \subset \mathcal{J}(\Gamma)$ . Contradiction.

From the assumption, there exists an integer  $k > 1$  such that  $f^k(K) \subset \mathcal{G}$ . Then  $f^{k-1}(K) \subset \mathcal{G} \cup \mathcal{D}$  by Lemma 5.6 and hence  $f^{k-1}(K) \subset \mathcal{G}$  by the assumption. Therefore  $f^n(K) \subset \mathcal{G}$  for all  $n > 0$ . It follows that  $K$  is eventually periodic by Lemma 5.6. It is a contradiction. Now the claim is proved.

By the claim, there exists an integer  $m > 0$  and a component  $W$  of  $\mathcal{D} \cup \mathcal{Q}$  such that  $f^m(K) \subset W$ . If  $W$  is a component of  $\mathcal{Q}$ , then  $f^{m+1}(K) \subset \mathcal{D} \cup \mathcal{Q}$  since  $W$  is disjoint from  $f^{-1}(\mathcal{P}_f)$  and  $f^{-1}(\mathcal{R})$ . Applying Sullivan's no wandering Fatou domain Theorem for the renormalization of  $f$ , we know that there exists an integer  $n > m$  such that  $f^n(K) \subset \mathcal{D}$ .  $\square$

**Proposition 5.12.** *Each wandering component of  $\mathcal{K}(\Gamma)$  is a single point.*

*Proof.* For each simply-connected domain  $D \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$ , denote by

$$\text{h-diameter}(D) = \sup_{z, w \in D} \ell[\gamma(z, w)],$$

where  $\gamma(z, w)$  is an arc in  $D$  connecting the points  $z$  and  $w$ , and  $\ell[\gamma(z, w)]$  is the infimum of the length of arcs in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$  under the orbifold metric over all the arcs in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$  homotopic to  $\gamma(z, w)$  rel  $\mathcal{P}_f \cup \{z, w\}$ .

It is easy to check that if  $\overline{D}$  is locally connected and disjoint from super-attracting periodic points of  $f$ , then  $\text{h-diameter}(D) < \infty$ .

For each component  $D$  of  $\mathcal{D}$ ,  $\overline{D}$  is locally connected by Corollary 5.5 and disjoint from super-attracting periodic points of  $f$ . Thus  $\text{h-diameter}(D) < \infty$ . Therefore there exists a constant  $M < \infty$  such that  $\text{h-diameter}(D) \leq M$  for each component  $D$  of  $\mathcal{D}$ .

Let  $K$  be a wandering component of  $\mathcal{K}_s$ . From Lemma 5.11, there exists an infinite increasing sequence  $\{n_k\}$  of positive integers such that  $f^{n_k}(K) \subset D_k$  and  $D_k$  are components of  $\mathcal{D}$ . Let  $W_k$  be the component of  $f^{-n_k}(D_k)$  that contains  $K$ , then  $\text{h-diameter}(W_k) \leq M\lambda^{-k}$  since  $f$  is expanding under the orbifold metric, where  $\lambda > 1$  is a constant such that  $\|f'\| \geq \lambda$  on  $f^{-1}(\mathcal{D})$ . Thus the diameter of  $W_k$  converges to zero as  $k \rightarrow \infty$ . It follows that  $K$  is a single point.  $\square$

*Proof of Theorem 1.3.* Combining Propositions 5.3, 5.10, 5.12 and Theorem 5.4, we obtain Theorem 1.3.  $\square$

## 6 Coding of the dynamics

We will prove Theorem 1.4 in this section. At first we give some definitions.

**Definition 4.** A **dendrite** is a locally connected and uniquely arcwise connected continuum. Let  $\mathcal{T} \subset \widehat{\mathbb{C}}$  be a dendrite. A continuous onto map  $\tau : \mathcal{T} \rightarrow \mathcal{T}$  is called a **finite dendrite map** if there exists a finite tree  $T_0 \subset \mathcal{T}$  such that the following statements hold.

- (1) For each  $n \geq 1$ ,  $T_n := \tau^{-n}(T_0)$  is also a finite tree with  $v(T_n) = \tau^{-n}(v(T_0))$ , where  $v(\cdot)$  denotes the set of vertices of a tree.
- (2)  $T_n \subset T_{n+1}$  and  $v(T_n) \subset v(T_{n+1})$ .
- (3)  $\tau$  is a homeomorphism on each edge of  $T_n$ .
- (4)  $\cup_{n \geq 0} T_n$  is dense in  $\mathcal{T}$ .
- (5)  $\deg \tau := \sup\{\#\tau^{-1}(x), x \in \mathcal{T}\} < \infty$ .

### 6.1 The tower of tree maps

**Definition 5.** By a **tower of tree maps** we mean an infinite sequence of triples  $\{T_n, \iota_n, \tau_n\}_{n \geq 0}$ , where  $T_n$  are finite trees,  $\iota_n : T_n \rightarrow T_{n+1}$  are inclusions and  $\tau_n : T_{n+1} \rightarrow T_n$  are continuous onto maps such that:

- (1)  $\iota_n(\mathcal{V}_n) \subset \mathcal{V}_{n+1}$ , where  $\mathcal{V}_n$  is the set of vertices of  $T_n$ ;



- (2)  $\tau_n^{-1}(\mathcal{V}_n) = \mathcal{V}_{n+1}$ ; and  
 (3) the following diagram commutes:

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{\tau_n} & T_n & \xrightarrow{\tau_{n-1}} & T_{n-1} & \xrightarrow{\tau_{n-2}} & \cdots \xrightarrow{\tau_1} T_1 \xrightarrow{\tau_0} T_0 \\
 & & \downarrow \iota_n & & \downarrow \iota_{n-1} & & \downarrow \iota_1 & & \downarrow \iota_0 \\
 \cdots & \xrightarrow{\tau_{n+1}} & T_{n+1} & \xrightarrow{\tau_n} & T_n & \xrightarrow{\tau_{n-1}} & \cdots \xrightarrow{\tau_2} T_2 \xrightarrow{\tau_1} T_1 \xrightarrow{\tau_0} T_0
 \end{array}$$

The degree of the tree map  $\tau_n : T_{n+1} \rightarrow T_n$  is defined by

$$\deg \tau_n = \sup \{ \# \tau_n^{-1}(y), y \in T_n \}.$$

Note that the sequence  $\{\deg \tau_n\}$  is increasing. The degree of the tower is defined to be its limit as  $n \rightarrow \infty$ .

A tower of tree maps  $\{T_n, \iota_n, \tau_n\}_{n \geq 0}$  is called **expanding** if there exist a constant  $\lambda > 1$  and a linear metric on  $T_0$  such that  $(\tau_0 \circ \iota_0) : T_0 \rightarrow T_0$  is  $C^1$  under this linear metric and the norm of its derivative is bigger than  $\lambda$  on  $T_0$ .

**Theorem 6.1.** *Let  $\{T_n, \iota_n, \tau_n\}_{n \geq 0}$  be an expanding tower of tree maps. Suppose that its degree is bounded. Then there exist an expanding finite dendrite map  $\tau : \mathcal{T} \rightarrow \mathcal{T}$  and inclusions  $i_n : T_n \rightarrow \mathcal{T}$  for all  $n \geq 0$  such that  $i_n = i_{n+1} \circ \iota_n$  and the following diagram commutes:*

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{\tau_{n+1}} & T_{n+1} & \xrightarrow{\tau_n} & T_n & \xrightarrow{\tau_{n-1}} & \cdots \xrightarrow{\tau_1} T_1 \xrightarrow{\tau_0} T_0 \\
 & & \downarrow i_{n+1} & & \downarrow i_n & & \downarrow i_1 & & \downarrow i_0 \\
 \cdots & \xrightarrow{\tau} & \mathcal{T} & \xrightarrow{\tau} & \mathcal{T} & \xrightarrow{\tau} & \cdots \xrightarrow{\tau} \mathcal{T} \xrightarrow{\tau} \mathcal{T}
 \end{array}$$

Moreover the finite dendrite map  $\tau : \mathcal{T} \rightarrow \mathcal{T}$  is unique up to topological conjugacy.

We will call  $\tau : \mathcal{T} \rightarrow \mathcal{T}$  the **limit** of the tower of tree maps  $\{T_n, \iota_n, \tau_n\}_{n \geq 0}$ .

*Proof.* Let  $|\cdot|$  be an expanding linear metric on  $T_0$ , i.e. there exists a constant  $\lambda > 1$  such that  $|(\tau_0 \circ \iota_0)'| > \lambda$  on  $T_0$ . Define a metric on  $\iota_0(T_0) \subset T_1$  such that  $\iota_0$  is an isometry and a metric on  $T_1 \setminus \iota_0(T_0)$  such that the norm of the derivative of  $\tau_0$  is a constant  $\lambda_1 > \max\{\lambda, d\}$ , where  $d$  is the degree of the tower. Then we get a metric on  $T_1$  such that  $\iota_0$  is an isometry and  $\lambda_1 \geq |\tau'_0| \geq \lambda$  on  $T_1$ .

Inductively, we can define a metric on each  $T_n$  with  $n \geq 1$  such that  $\iota_{n-1}$  is an isometry and  $\lambda_1 \geq |\tau'_{n-1}| \geq \lambda$  on  $T_n$ .

Denote by  $\tilde{\mathcal{S}}$  the space consisting of left infinite sequences  $(\cdots, t_1, t_0)$  with  $t_n \in T_{n+k}$  for some integer  $k \geq 0$ , such that for any  $n \geq 0$ , if  $t_n \in T_{n+k}$ , then  $t_{n+1} = \iota_{n+k}(t_n)$ . Define an equivalent relation on  $\tilde{\mathcal{S}}$  by

$$(\cdots, s_1, s_0, ) \sim (\cdots, t_1, t_0)$$

if there exists an integer  $k$  such that  $s_n = t_{n+k}$  whenever  $n, n+k \geq 0$ . Let  $\mathcal{S}$  be the quotient space  $\tilde{\mathcal{S}}/\sim$ . For each point  $(\cdots, t_1, t_0)$  in  $\mathcal{S}$ , we denote by  $[\cdots, t_1, t_0]$  representing

its equivalence class in  $\mathcal{S}$ . Since  $\iota_n$  is an isometry, there exists a metric  $\rho$  on  $\mathcal{S}$  such that the inclusion  $i_n : T_n \rightarrow \mathcal{S}$  defined by:

$$i_n(t) = [\cdots, \iota_{n+1} \circ \iota_n(t), \iota_n(t), t]$$

is an isometry. Clearly,  $i_{n+1} \circ \iota_n = i_n$  on  $T_n$ .

Since  $\tau_n \circ \iota_n = \iota_{n-1} \circ \tau_{n-1}$  on  $T_n$ , there exists a continuous onto map  $\tau : \mathcal{S} \rightarrow \mathcal{S}$  such that  $\tau \circ i_n = i_{n-1} \circ \tau_{n-1}$  on  $T_n$ . Moreover  $\lambda_1 \geq |\tau'| \geq \lambda$ .

Since  $\lambda_1 > d$ ,  $\mathcal{S}$  is bounded. Let  $\mathcal{T}$  be the completion of  $\mathcal{S}$ . Then it is a dendrite. The map  $\tau$  can be extended to be a continuous onto map on  $\mathcal{T}$  since  $\tau$  is uniformly continuous.

The proof of the uniqueness of  $\tau : \mathcal{T} \rightarrow \mathcal{T}$  is direct.  $\square$

## 6.2 Coding of the dynamics

Let  $f$  be a post-critically finite rational map with a stable Cantor multicurve  $\Gamma$ . We want to define a tower of tree maps from  $f$  and construct a semi-conjugacy from  $f$  to its limit.

Denote by  $\Gamma_n$  the collection of the curves in  $f^{-n}(\Gamma)$  for  $n \geq 0$ . Then each curve in  $\Gamma_n$  is essential in  $\widehat{\mathbb{C}} \setminus f^{-n}(\mathcal{P}_f)$  and no two of them are homotopic in  $\widehat{\mathbb{C}} \setminus f^{-n}(\mathcal{P}_f)$ .

For each curve  $\gamma \in \Gamma_n$ , denote by  $\Gamma_{n+1}(\gamma)$  the curves in  $\Gamma_{n+1}$  homotopic to  $\gamma$  rel  $f^{-n}(\mathcal{P}_f)$ . Since  $\Gamma$  is pre-stable and stable, the next lemma is easy to check:

**Lemma 6.2.** *For each curve  $\gamma \in \Gamma_n$ ,  $\Gamma_{n+1}(\gamma) \neq \emptyset$  and any curve in  $\Gamma_{n+1} \setminus \Gamma_{n+1}(\gamma)$  does not separate curves in  $\Gamma_{n+1}(\gamma)$ . Moreover, for any two curves  $\gamma_1$  and  $\gamma_2$  in  $\Gamma_n$ , if there is no curve in  $\Gamma_n$  separating  $\gamma_1$  from  $\gamma_2$ , then each curve in  $\Gamma_{n+1} \setminus (\Gamma_{n+1}(\gamma_1) \cup \Gamma_{n+1}(\gamma_2))$  does not separate curves in  $\Gamma_{n+1}(\gamma_1) \cup \Gamma_{n+1}(\gamma_2)$ .*

**Dual trees.** For any  $n \geq 0$ , let  $T_n$  be the dual tree of  $\Gamma_n$  defined by the following: There is a bijection between vertices of  $T_n$  and components of  $\widehat{\mathbb{C}} \setminus \Gamma_n$ . Two vertices are connected by an edge if their corresponding components of  $\widehat{\mathbb{C}} \setminus \Gamma_n$  have a common boundary component, which is a curve in  $\Gamma_n$ . Thus there is a bijection between edges of  $T_n$  and curves in  $\Gamma_n$ . Denote by  $e_\gamma$  the edge of  $T_n$  corresponding to the curve  $\gamma \in \Gamma_n$ .

**Inclusion maps.** The homotopy rel  $f^{-n}(\mathcal{P}_f)$  induces an inclusion  $\iota_n : T_n \rightarrow T_{n+1}$  by the following: For each curve  $\gamma \in \Gamma_n$ , define

$$\iota_n : e_\gamma \rightarrow \bigcup_{\beta \in \Gamma_{n+1}(\gamma)} e_\beta \cup \{\text{common endpoints of } e_\beta\}$$

to be a homeomorphism such that it preserves the orientation induced by the orientation on these curves.

By the definition,  $\iota_n$  is continuous on each edge. The continuity of  $\iota_n$  at vertices comes from Lemma 6.2. The injectivity comes from the fact that no two curves in  $\Gamma_n$  are homotopic rel  $f^{-n}(\mathcal{P}_f)$ .

**Induced tree maps.** Given any  $n \geq 0$ . A continuous map  $\tau : T_{n+1} \rightarrow T_n$  is called an induced tree map if for each edge  $e_\gamma$  of  $T_{n+1}$  corresponding to a curve  $\gamma \in \Gamma_{n+1}$ ,

$\tau : e_\gamma \rightarrow e_{f(\gamma)}$  is a homeomorphism such that it preserves the orientation induced by the map  $f : \gamma \rightarrow f(\gamma)$ . It is easy to check that induced tree maps always exist.

**Lemma 6.3.** *There exists a linear metric  $\rho_1$  on the tree  $T_1$  and an induced tree map  $\tau_0 : T_1 \rightarrow T_0$  such that  $\iota_0 \circ \tau_0$  is linear on each edge of  $T_1$  and  $|(\iota_0 \circ \tau_0)'| \geq \lambda$  for some constant  $\lambda > 1$ .*

*Proof.* Let  $\{e_1, \dots, e_n\}$  be the edges of  $T_0$ . Let  $M = (b_{ij})$  be the reduced transition matrix of  $\Gamma$  defined in §2. Then its leading eigenvalue  $\lambda_0 > 1$  by Lemma 2.4 since  $\Gamma$  is a Cantor multicurve. Thus there exist a constant  $\lambda \in (1, \lambda_0)$  and a positive eigenvector  $\mathbf{v} = (v(e_i))$  such that  $M\mathbf{v} > \lambda\mathbf{v}$  by Lemma A.1 in [6]. Define a linear metric  $\rho_1$  on  $T_1$  such that for each edge  $e$  of  $T_1$ , it has length  $v(\tau_0(e))$ . Then the length of  $\iota_0(e_i)$  is:

$$|\iota_0(e_i)| = \sum_j b_{ij} v(e_j) > \lambda v(e_i).$$

Define  $\tau_0 : T_1 \rightarrow T_0$  to be an induced tree map such that  $\iota_0 \circ \tau_0$  is linear on each edge of  $T_1$ . Then  $|(\iota_0 \circ \tau_0)'| > \lambda$ .  $\square$

There exists an induced tree map  $\tau_1 : T_2 \rightarrow T_1$  such that  $\tau_1 \circ \iota_1 = \iota_0 \circ \tau_0$  on  $T_1$ . Inductively, for each  $n \geq 2$ , there exists an induced tree map  $\tau_{n-1} : T_n \rightarrow T_{n-1}$  such that  $\tau_{n-1} \circ \iota_{n-1} = \iota_{n-2} \circ \tau_{n-2}$  on  $T_{n-1}$ . Then  $\{T_n, \iota_n, \tau_n\}_{n \geq 0}$  is an expanding tower of tree maps with degree  $\deg \tau_n \leq \deg f$ . We call it the **induced tower of tree maps** of  $f$  with respect to the multicurve  $\Gamma$ .

Denote by  $\tau_f : \mathcal{T}(\Gamma) \rightarrow \mathcal{T}(\Gamma)$  the limit of the induced tower of tree maps of  $f$  with respect to the multicurve  $\Gamma$ . Then it is an expanding finite dendrite map by Theorem 6.1. The next theorem is a more precise version of Theorem 1.4.

**Theorem 6.4.** *Let  $f$  be a post-critically rational map with a stable Cantor multicurve  $\Gamma$ . Then there exist an expanding finite dendrite map  $\tau_f : \mathcal{T}(\Gamma) \rightarrow \mathcal{T}(\Gamma)$  and a continuous onto map  $\Theta : \widehat{\mathbb{C}} \rightarrow \mathcal{T}(\Gamma)$  such that  $\tau_f \circ \Theta = \Theta \circ f$ . Moreover, for each point  $t \in \mathcal{T}(\Gamma)$ , the fiber  $\Theta^{-1}(t)$  is a component of  $\mathcal{J}(\Gamma)$  or  $\mathcal{K}(\Gamma)$ .*

*Proof.* Let  $\{T_n, \iota_n, \tau_n\}_{n \geq 0}$  be the induced tower of tree maps of  $f$  with respect to  $\Gamma$ . We may identify  $T_n$  with  $i_n(T_n) \subset \mathcal{T}(\Gamma)$  by Theorem 6.1. Then  $\tau_n = \tau_f$ . Let  $\mathcal{I} \subset T_1$  be the union of open edges of  $T_1$  contained in  $T_0$ . Let  $\sigma$  be the restriction of  $\tau_f$  on  $\mathcal{I}$ . Then  $\sigma$  is an expanding linear system. So  $\mathcal{J}_\sigma$  is dense in  $\mathcal{I}$ . Let  $g = f : \mathcal{A}^1 \rightarrow \mathcal{A}$  be the exact annular system obtained in Theorem 4.1. Then there exists a bijection from the components of  $\mathcal{A}^1$  to the components of  $\mathcal{I}$  according to the correspondence from  $\Gamma_1$  to the edges of  $T_1$  and the homotopy rel  $f^{-1}(\mathcal{P}_f)$ .

Define a map  $\Theta_0 : \mathcal{J}_g \rightarrow \mathcal{J}_\sigma$  by the itinerary as in Proposition 3.5. It is order-preserving. Since its image  $\mathcal{J}_\sigma$  is dense in  $\mathcal{I}$ , it can be extended to a continuous onto map  $\Theta_0 : \widehat{\mathbb{C}} \rightarrow T_0$  such that each component of  $\widehat{\mathbb{C}} \setminus \mathcal{A}$  maps to a vertex of  $T_0$ . It is easy to check that  $\tau_f \circ \Theta_0 = \Theta_0 \circ f$  on  $\mathcal{A}^1$ .

Pullback the map  $\Theta_0$  by the above equation, we get a continuous onto map  $\Theta_1 : \widehat{\mathbb{C}} \rightarrow T_1$  such that each component of  $\widehat{\mathbb{C}} \setminus f^{-1}(\mathcal{A})$  maps to a vertex of  $T_1$  and  $\tau_f \circ \Theta_1 = \Theta_1 \circ f$  on  $f^{-1}(\mathcal{A})$ .

Inductively, we get a sequence of continuous maps  $\Theta_n : \widehat{\mathbb{C}} \rightarrow T_n$  such that each component of  $\widehat{\mathbb{C}} \setminus f^{-n}(\mathcal{A})$  maps to a vertex of  $T_n$  and  $\tau_f \circ \Theta_n = \Theta_n \circ f$  on  $f^{-n}(\mathcal{A})$ . It is easy to check that  $\Theta_n$  converges uniformly to a continuous onto map  $\Theta : \widehat{\mathbb{C}} \rightarrow \mathcal{T}(\Gamma)$  as  $n \rightarrow \infty$  and the map  $\Theta$  satisfies all the conditions.  $\square$

## 7 Wandering continua

**Definition 6.** Let  $f$  be a rational map. By a **wandering continuum** we mean a non-degenerate continuum  $K \subset \mathcal{J}_f$  (i.e.  $K$  is a connected compact set consisting of more than one point) such that  $f^n(K) \cap f^m(K) = \emptyset$  for any  $n > m \geq 0$ .

The existence of wandering continua for polynomials has been studied by many authors (refer to [3, 12, 13, 14, 26]). It is proved that for a polynomial without irrational indifferent periodic cycles, there is no wandering continuum on the Julia set if and only if the Julia set is locally connected [3, 12].

A continuum  $E \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$  is called **essential** if there are exactly two components of  $\widehat{\mathbb{C}} \setminus E$  containing points of  $\mathcal{P}_f$  and each of them contains at least two points of  $\mathcal{P}_f$ .

**Lemma 7.1.** Let  $f$  be a post-critically finite rational map. Suppose that  $K \subset \mathcal{J}_f$  is a wandering continuum. Then either  $f^n(K)$  is 1-connected for all  $n \geq 0$ ; or there exists an integer  $N \geq 0$  such that  $f^n(K)$  is essential for  $n \geq N$ .

*Proof.* Set  $K_n = f^n(K)$  for  $n \geq 0$ . Since  $\#\mathcal{P}_f < \infty$  and  $K$  is wandering, we have  $K_n \cap \mathcal{P}_f = \emptyset$  for all  $n \geq 0$ . Thus if  $K_n$  is 1-connected, then  $K_m$  is also 1-connected for  $m \leq n$ .

Suppose that there is an integer  $n_0 \geq 1$  such that  $K_{n_0}$  is not 1-connected, then  $K_n$  is not 1-connected for all  $n \geq n_0$ . Let  $s(K_n) \geq 1$  be the number of components of  $\widehat{\mathbb{C}} \setminus K_n$  containing points of  $\mathcal{P}_f$ . Since  $K_n$  are pairwise disjoint, there are at most  $(\#\mathcal{P}_f - 2)$  continua  $K_n$  with  $s(K_n) \geq 3$ . Thus there is an integer  $n_1 \geq n_0$  such that  $s(K_n) \leq 2$  for all  $n \geq n_1$ .

If  $s(K_n) \equiv 1$  for all  $n \geq n_1$ , let  $\widehat{K}_n$  be the union of  $K_n$  together with the components of  $\widehat{\mathbb{C}} \setminus K_n$  disjoint from  $\mathcal{P}_f$ , then  $f : \widehat{K}_n \rightarrow \widehat{K}_{n+1}$  is a homeomorphism for  $n \geq n_1$ . Since  $K_{n_1}$  is not 1-connected,  $\widehat{K}_{n_1} \setminus K_{n_1}$  is non-empty. Let  $U$  be a component of  $\widehat{K}_{n_1} \setminus K_{n_1}$ . Then  $U \cap \mathcal{P}_f = \emptyset$ . If  $U \cap \mathcal{J}_f \neq \emptyset$ , then  $f^m(U) \supset \mathcal{J}_f$  for some  $m \geq 1$ . But  $f^m(U)$  is a component of  $\widehat{\mathbb{C}} \setminus K_{n_1+m}$ . It is a contradiction. So  $U \cap \mathcal{J}_f = \emptyset$ . Noticing that  $\partial U \subset K_{n_1} \subset \mathcal{J}_f$ , the simply-connected domain  $U$  is exactly a Fatou domain. But  $\partial U$  is wandering. Thus it is a contradiction since there is no wandering Fatou domain by Sullivan's theorem (refer to [19]). Therefore there is an integer  $n_2 \geq n_1$  such that  $s(K_{n_2}) = 2$ .

We claim that  $s(K_n) \equiv 2$  for all  $n \geq n_2$ . Otherwise, assume that there is an integer  $m > n_2$  such that  $s(K_m) = 1$ , then there is a disk  $D$  containing  $K_m$  such that  $D \cap \mathcal{P}_f = \emptyset$ . Let  $D_n$  be the component of  $f^{n-m}(D)$  containing  $K_n$  for  $n_2 \leq n \leq m$ . Then  $D_n$  is disjoint from  $\mathcal{P}_f$ . So  $s(K_n) = 1$  for  $n_2 \leq n \leq m$ . This contradicts  $s(K_{n_2}) = 2$ .

We may assume  $\#\mathcal{P}_f \geq 3$  (otherwise  $f$  is conjugate to the map  $z \rightarrow z^{\pm d}$  and hence has no wandering continuum), then  $f$  has at most one exceptional point. If there is an integer  $m \geq n_2$  such that  $\widehat{\mathbb{C}} \setminus K_m$  has a component containing exactly one  $\mathcal{P}_f$  point, then there is a disk  $D \supset K_m$  such that  $D$  contains exactly one  $\mathcal{P}_f$  point. Let  $D_n$  be the component of  $f^{n-m}(D)$  containing  $K_n$  for  $n_2 \leq n \leq m$ . Then  $D_n$  is simply-connected and contains at most one point of  $\mathcal{P}_f$ . Thus  $\widehat{\mathbb{C}} \setminus K_n$  has a component containing exactly one  $\mathcal{P}_f$  point for  $n_2 \leq n \leq m$ . Therefore either there exists an integer  $N \geq n_2$  such that for  $n \geq N$ ,  $f^n(K)$  is essential, or  $\widehat{\mathbb{C}} \setminus f^n(K)$  has a component containing exactly one  $\mathcal{P}_f$  point for all  $n \geq n_2$ .

In the latter case, denote by  $U$  the component of  $\widehat{\mathbb{C}} \setminus K_{n_2}$  containing exactly one  $\mathcal{P}_f$  point. If  $U \cap \mathcal{J}_f \neq \emptyset$ , then there is an integer  $k > 0$  such that  $\widehat{\mathbb{C}} \setminus f^k(U)$  contains at most one point (an exceptional point). On the other hand, there is a disk  $D \supset K_{n_2+k}$  such that  $D$  contains exactly one  $\mathcal{P}_f$  point. Let  $D_{n_2}$  be the component of  $f^{-k}(D)$  containing  $K_{n_2}$ . Then  $D_{n_2}$  is simply-connected and contains at most one point of  $\mathcal{P}_f$ . Thus  $U \subset D_{n_2}$ . Therefore  $f^k(U) \subset D$  and hence  $\widehat{\mathbb{C}} \setminus D \subset \widehat{\mathbb{C}} \setminus f^k(U)$  contains at most one point. This contradicts  $\#\mathcal{P}_f \geq 3$ . So  $U$  is disjoint from  $\mathcal{J}_f$  and hence is a simply-connected Fatou domain. This again contradicts Sullivan's no wandering Fatou domain theorem.  $\square$

**Lemma 7.2.** *Suppose that  $K \subset \mathcal{J}_f$  is a wandering continuum and is not 1-connected. Then there is a multicurve  $\Gamma_K$  such that:*

- (1) *for each curve  $\gamma$  in  $\Gamma_K$ , there are infinitely many continua  $f^n(K)$  homotopic to  $\gamma$  rel  $\mathcal{P}_f$ , and*
- (2) *there is an integer  $N_1 \geq 0$  such that for  $n \geq N_1$ ,  $f^n(K)$  is essential and homotopic rel  $\mathcal{P}_f$  to a curve in  $\Gamma_K$ .*

*Proof.* By Lemma 7.1, there is an integer  $N \geq 0$  such that  $f^n(K)$  is essential for  $n \geq N$ . Since the  $f^n(K)$  are pairwise disjoint, for any integer  $m \geq N$ , we may choose an essential Jordan curve  $\beta_n$  in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$  for  $N \leq n \leq m$  such that they are pairwise disjoint and  $f^n(K)$  is homotopic to  $\beta_n$  rel  $\mathcal{P}_f$ . Let  $\Gamma_m$  be the collection of these curves. Let  $\tilde{\Gamma}_m \subset \Gamma_m$  be a multicurve such that each curve in  $\Gamma_m$  is homotopic to a curve in  $\tilde{\Gamma}_m$ . Then each curve in  $\tilde{\Gamma}_m$  is homotopic to a curve in  $\tilde{\Gamma}_{m+1}$ . This implies that  $\#\tilde{\Gamma}_m$  is increasing and hence there is an integer  $m_0 \geq N$  such that  $\#\tilde{\Gamma}_m$  is a constant for  $m \geq m_0$  since any multicurve contains at most  $\#\mathcal{P}_f - 3$  curves. Therefore each curve in  $\tilde{\Gamma}_{m+1}$  is homotopic to a curve in  $\tilde{\Gamma}_m$  for  $m \geq m_0$ . This shows that the multicurves  $\tilde{\Gamma}_m$  are homotopic to each other for all  $m \geq m_0$ .

Let  $\Gamma_K \subset \tilde{\Gamma}_{m_0}$  be the sub-collection consisting of curves  $\gamma \in \tilde{\Gamma}_{m_0}$  such that there are infinitely many  $f^n(K)$  homotopic to  $\gamma$  rel  $\mathcal{P}_f$ . Then it is non-empty and hence is a multicurve. Obviously,  $\Gamma_K$  is uniquely determined by  $K$  and there is an integer  $N_1 \geq 0$  such that for  $n \geq N_1$ ,  $f^n(K)$  is essential and homotopic rel  $\mathcal{P}_f$  to a curve in  $\Gamma_K$ .  $\square$

**Lemma 7.3.**  *$\Gamma_K$  is an irreducible Cantor multicurve.*

*Proof.* By Lemma 7.2, there exists an integer  $N_1 \geq 0$  such that  $f^n(K)$  for  $n \geq N_1$  is homotopic to a curve in  $\Gamma_K$  rel  $\mathcal{P}_f$ . Thus  $\Gamma_K$  is pre-stable. For any pair  $(\gamma, \alpha) \in \Gamma_K \times \Gamma_K$ , there are integers  $k_2 > k_1 \geq N_1$  such that  $f^{k_1}(K)$  is homotopic to  $\gamma$  and  $f^{k_2}(K)$  is homotopic to  $\alpha$  rel  $\mathcal{P}_f$ . Thus  $f^{k_1-k_2}(\alpha)$  has a component  $\delta$  homotopic to  $\gamma$  rel  $\mathcal{P}_f$ . So for

$1 < i < k_2 - k_1$  the curve  $f^i(\delta)$  is homotopic rel  $\mathcal{P}_f$  to  $f^{k_1+i}(K)$  and hence to a curve in  $\Gamma_K$  rel  $\mathcal{P}_f$ . This shows that  $\Gamma_K$  is irreducible.

Now we want to prove that  $\Gamma_K$  is a Cantor multicurve. We may apply Lemma 2.3 and assume by contradiction that  $f^{-1}(\gamma)$  for each  $\gamma \in \Gamma_K$  has exactly one component homotopic rel  $\mathcal{P}_f$  to a curve in  $\Gamma_K$ .

Assume  $N_1 = 0$  for simplicity. Denote by  $\Gamma_K = \{\gamma_0, \dots, \gamma_{p-1}\}$  such that  $\gamma_0$  is homotopic to  $K$  and  $\gamma_n$  is homotopic to a component of  $f^{-1}(\gamma_{n+1})$  for  $0 \leq n < p$  (set  $\gamma_p = \gamma_0$ ). It makes sense since for each  $\gamma \in \Gamma_K$ ,  $f^{-1}(\gamma)$  has exactly one component homotopic rel  $\mathcal{P}_f$  to a curve in  $\Gamma_K$ . Then  $f^n(K)$  is homotopic to  $\gamma_k$  if  $n \equiv k \pmod{p}$ .

For  $n \geq 0$  and  $k \geq 1$  denote by  $A(n, n + kp)$  the unique annular component of  $\widehat{\mathbb{C}} \setminus (f^n(K) \cup f^{n+kp}(K))$ . Then  $f^m : A(n, n + kp) \rightarrow A(n + m, n + kp + m)$  is proper for any  $m \geq 1$ . This is because that  $A(n + m, n + kp + m)$  is disjoint from  $\mathcal{P}_f$  and homotopic to  $f^{n+m}(K)$ , so  $f^{-m}(A(n + m, n + kp + m))$  has a unique component homotopic to  $f^n(K)$ , which must be  $A(n, n + kp)$ .

One may choose  $(n, k)$  such that  $A(n, n + kp)$  contains points of  $\mathcal{J}_f$ . On the other hand,  $f^m$  is proper on  $A(n, n + kp)$  for all  $m \geq 1$ , whose image contains no points of  $\mathcal{P}_f$ . It is a contradiction.  $\square$

*Proof of Theorem 1.2.* Suppose that  $K \subset \mathcal{J}_f$  is a wandering continuum and is not 1-connected. Then  $\Gamma_K$  is an irreducible Cantor multicurve by Lemma 7.3. By Lemma 7.2, there exists an integer  $N_1 \geq 0$  such that  $f^n(K)$  for  $n \geq N_1$  is homotopic to a curve in  $\Gamma_K$  rel  $\mathcal{P}_f$ . We assume  $N_1 = 0$  for simplicity.

Let  $\mathcal{E}$  be the collection of the essential components  $E$  of  $f^{-m}(f^n(K))$  for  $n, m \geq 0$  such that  $f^i(E)$  is homotopic to a curve in  $\Gamma_K$  for  $0 \leq i < m$ . Then  $f(E) \in \mathcal{E}$  for any element  $E \in \mathcal{E}$ , and any two elements in  $\mathcal{E}$  are either disjoint or one contains another as subsets of  $\widehat{\mathbb{C}}$ .

For each  $\gamma \in \Gamma_K$ , let  $\mathcal{E}(\gamma)$  be the sub-collection of continua in  $\mathcal{E}$  homotopic to  $\gamma$  rel  $\mathcal{P}_f$ . We claim that for any continuum  $E \in \mathcal{E}(\gamma)$ , there are two disjoint continua  $E_1, E_2 \in \mathcal{E}(\gamma)$  such that  $E \subset A(E_1, E_2)$ , where  $A(E_1, E_2)$  denotes the unique annular component of  $\widehat{\mathbb{C}} \setminus (E_1 \cup E_2)$ .

Consider  $\{f^n(E)\}$  for  $0 \leq n \leq 2 \cdot \#\Gamma_K + 1$ . There is a curve  $\beta \in \Gamma_K$  such that at least three of them are contained in  $\mathcal{E}(\beta)$ . Numerate them by  $f^{n_i}(E)$  ( $i = 1, 2, 3$ ) such that  $f^{n_3}(E) \subset A(f^{n_1}(E), f^{n_2}(E))$ . Let  $A$  be the component of  $f^{-n_3}(A(f^{n_1}(E), f^{n_2}(E)))$  that contains  $E$ . Then  $A = A(E_1, E_2)$  where  $E_i$  ( $i = 1, 2$ ) is a component of  $f^{-n_3}(f^{n_i}(E))$ . Now the claim is proved.

Denote  $A(\gamma) = \cup A(E, E')$  for all disjoint pairs  $E, E' \in \mathcal{E}(\gamma)$ . Then  $A(\gamma)$  is an annulus in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$  homotopic to  $\gamma$  rel  $\mathcal{P}_f$ , and  $A(\gamma) \cap A(\beta) = \emptyset$  for distinct curves  $\beta, \gamma \in \Gamma_K$ .

Denote by  $\mathcal{A} = \cup_{\gamma \in \Gamma_K} A(\gamma)$  and  $\mathcal{A}^1$  the union of components of  $f^{-1}(\mathcal{A})$  homotopic to curves in  $\Gamma_K$ . Then  $\mathcal{A}^1 \subset \mathcal{A}$  and  $\partial \mathcal{A} \subset \partial \mathcal{A}^1$  by the claim and the definition of  $\mathcal{E}$ . So  $g = f|_{\mathcal{A}^1} : \mathcal{A}^1 \rightarrow \mathcal{A}$  is an exact annular system. In particular,  $f^n(K) \subset \mathcal{A}$  and hence  $f^n(K) \subset \mathcal{A}^1$  for all  $n \geq 0$ . So  $K \subset \mathcal{J}_g$ . But  $K$  is essential. Therefore  $K$  is a Jordan curve by Theorem 3.9.  $\square$



## 8 Foldings of polynomials

In this section, we introduce a topological surgery procedure to transform polynomials into rational maps with Cantor multicurves.

### 8.1 Folding maps

Let  $F$  be a post-critically finite branched covering of  $\widehat{\mathbb{C}}$  and  $\beta$  be an essential Jordan curve in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_F$ . The pair  $(F, \beta)$  is called a **folding map** if  $F^{-1}(\beta)$  contains at least two curves and each of them is homotopic to  $\beta$  rel  $\mathcal{P}_F$ . The curve  $\beta$  is called the **folding curve** and

$$m(F, \beta) := \#\{\text{components of } F^{-1}(\beta)\}$$

is called the **folding times**.

Two folding maps  $(F, \beta)$  and  $(G, \alpha)$  are called **Thurston equivalent** if  $F$  is Thurston equivalent to  $G$  through a pair of homeomorphisms  $(\phi, \psi)$  such that  $\phi(\beta)$  is homotopic to  $\alpha$  rel  $\mathcal{P}_G$ .

Let  $(F, \beta)$  be a folding map. Denote by  $U, V$  the two components of  $\widehat{\mathbb{C}} \setminus \beta$ . Denote by  $U_1, V_1$  the two disc components of  $\widehat{\mathbb{C}} \setminus F^{-1}(\beta)$  such that  $U_1$  is homotopic to  $U$  (i.e. there is an isotopy  $\theta$  of  $\widehat{\mathbb{C}}$  rel  $\mathcal{P}_F$  such that  $U_1 = \theta(U)$ ). Then  $V_1$  is homotopic to  $V$ . There are three possibilities:

Type A:  $F(U_1) = U$  and  $F(V_1) = U$ .

Type B:  $F(U_1) = U$  and  $F(V_1) = V$ .

Type C:  $F(U_1) = V$  and  $F(V_1) = U$ .

Define

$$d(F, \beta) = \begin{cases} \deg(F|_{U_1}), & \text{in type A,} \\ \min\{\deg(F|_{U_1}), \deg(F|_{V_1})\}, & \text{in type B,} \\ \sqrt{\deg(F|_{U_1}) \deg(F|_{V_1})}, & \text{in type C.} \end{cases}$$

The following facts are easy to check:

- $(F, \beta)$  is of type A if and only if  $m(F, \beta)$  is even.
- $(F^n, \beta)$  is also a folding map with  $m(F^n, \beta) = m(F, \beta)^n$  and  $d(F^n, \beta) = d(F, \beta)^n$  for  $n \geq 1$ .
- If  $(F, \beta)$  is of type A (resp. type B), then  $(F^n, \beta)$  is also of type A (resp. type B) for  $n \geq 1$ ; If  $(F, \beta)$  is of type C, then  $(F^{2k-1}, \beta)$  is of type C and  $(F^{2k}, \beta)$  is of type B for  $k \geq 1$ .

We will prove the following theorems in this section.

**Theorem 8.1.** *Let  $(F, \beta)$  be a folding map. Suppose that*

- (a) *the multicurve  $\{\beta\}$  is not a Thurston obstruction;*
- (b) *any stable multicurve disjoint from  $\beta$  is not a Thurston obstruction; and*
- (c)  *$d(F, \beta) < m(F, \beta)$ .*

*Then  $F$  has no Thurston obstruction.*

**Theorem 8.2.** *Let  $(F, \beta)$  be a folding map. Suppose that*

- (a) the multicurve  $\{\beta\}$  is not a Thurston obstruction;*
- (b) any stable multicurve disjoint from  $\beta$  is not a Thurston obstruction; and*
- (c') there exist an integer  $p \geq 1$  and a finite tree  $T \subset \widehat{\mathbb{C}} \setminus \beta$  whose vertices are contained in  $\mathcal{P}_F$  such that  $F^p(T)$  is contained in  $T$  homotopically (i.e., there exists a homeomorphism  $\theta$  of  $\widehat{\mathbb{C}}$  isotopic to the identity rel  $\mathcal{P}_F$  such that  $F^p(T) \subset \theta(T)$ ),  $F^p$  is injective on  $T$  and  $F^{-p}(T)$  has a component homotopic to  $\beta$  rel  $\mathcal{P}_F$ .*

*Then  $F$  has no Thurston obstruction.*

**Remark.** (1) Obviously the conditions (a) and (b) in the theorems are necessary. We will give examples in §7.5 to show that these two conditions are not sufficient, and the conditions (c) and (c') are not necessary.

(2) Denote  $m = m(F, \beta)$  and  $d_i$  ( $1 \leq i \leq m$ ) the degree of  $F$  on the components of  $F^{-1}(\beta)$ . Then the multicurve  $\{\beta\}$  is not a Thurston obstruction if and only if

$$\lambda(\{\beta\}) = \frac{1}{d_1} + \cdots + \frac{1}{d_m} < 1.$$

## 8.2 Foldings of polynomials

Let  $(F, \beta)$  be a folding map of type A and  $g$  be a polynomial with connected Julia set. We say  $(F, \beta)$  is the **folding of  $g$**  if it is Thurston equivalent to another folding map  $(G, \alpha)$  such that  $G^{-1}(U)$  has a disc component  $U_1 \Subset U$ , where  $U$  is one of two Jordan domains enclosed by  $\alpha$ , and  $G|_{U_1} = g$ .

Let  $(F, \beta)$  be a folding map of type B and  $(g_1, g_2)$  be a pair of polynomials with connected Julia sets. We say  $(F, \beta)$  is the **folding of  $(g_1, g_2)$**  if it is Thurston equivalent to another folding map  $(G, \alpha)$  such that there are disjoint Jordan domains  $U$  and  $V$  in  $\widehat{\mathbb{C}}$  with both  $\partial U$  and  $\partial V$  homotopic to  $\alpha$  rel  $\mathcal{P}_G$ , both  $G^{-1}(U)$  and  $G^{-1}(V)$  have a disc component  $U_1 \Subset U$  and  $V_1 \Subset V$ ,  $G|_{U_1} = g_1$  and  $G|_{V_1} = g_2$ .

The following result relates a folding map to a folding of polynomials, without taking into account whether the latter map is Thurston equivalent to a rational map or not.

**Theorem 8.3.** *Let  $(F, \beta)$  be a folding map of type A (or type B). Suppose that  $\{\beta\}$  is not a Thurston obstruction. Then  $(F, \beta)$  is the folding of a polynomial  $g$  (or a pair of polynomials  $(g_1, g_2)$ ) if and only if any stable multicurve disjoint from  $\beta$  is not a Thurston obstruction. Moreover, the polynomial  $g$  (or the pair of polynomials  $(g_1, g_2)$ ) is unique up to holomorphic conjugation.*

*Proof.* Suppose that  $(F, \beta)$  is the folding of a polynomial  $g$  whose Julia set is connected. By the definition, there is a folding map  $(G, \alpha)$  which is Thurston equivalent to  $(F, \beta)$ , such that  $G^{-1}(U)$  has a disc component  $U_1 \Subset U$ , where  $U$  is one of two Jordan domains enclosed by  $\alpha$ , and  $G|_{U_1} = g$ .

Let  $\Gamma$  be an irreducible multicurve of  $G$  which is disjoint from  $\alpha$ . If there is one curve  $\gamma \in \Gamma$  homotopic to  $\alpha$  rel  $\mathcal{P}_G$ , then  $\Gamma = \{\gamma\}$  and hence  $\lambda(\Gamma) < 1$ . Now we assume that for any  $\gamma \in \Gamma$ ,  $\gamma$  is not homotopic to  $\alpha$  rel  $\mathcal{P}_G$ . Then  $\gamma$  is homotopic rel  $\mathcal{P}_G$  to a curve in  $U$ .

Set  $\mathcal{P} = (\mathcal{P}_G \cap U) \cup \{\infty\}$ . Then  $\mathcal{P}_g \subset \mathcal{P}$ ,  $g(\mathcal{P}) \subset \mathcal{P}$  and  $\Gamma$  is a multicurve of the marked polynomial  $(g, \mathcal{P})$ . By Theorem 3.3 in [6],  $\lambda(\Gamma) < 1$ . Applying Lemma 2.2, we see that  $\lambda(\Gamma) < 1$  for any stable multicurve  $\Gamma$ .

Conversely, suppose that any stable multicurve of  $F$  disjoint from  $\beta$  is not a Thurston obstruction. Let  $W$  be the component of  $\widehat{\mathbb{C}} \setminus \beta$  such that  $F(W_1) = W$ , where  $W_1$  is the component of  $\widehat{\mathbb{C}} \setminus F^{-1}(\beta)$  homotopic to  $W$ . Then there is an isotopy  $\theta$  of  $\widehat{\mathbb{C}}$  rel  $\mathcal{P}_F$  such that  $\theta(W_1) \Subset W$ . Set  $G_1 = F \circ \theta^{-1}$ . Then  $(G_1, \beta)$  is Thurston equivalent to  $(F, \beta)$ .

Denote  $\mathcal{P} = \mathcal{P}_F \cap W$ . Then  $G_1(\mathcal{P}) \subset \mathcal{P}$  and  $G_1 : (\overline{\theta(W_1)}, \mathcal{P}) \rightarrow (\overline{W}, \mathcal{P})$  is a marked repelling system (ref to [6]). Applying Lemma 2.1 and Theorem 3.5 in [6], there exist a polynomial-like map  $g_1 : V_1 \rightarrow V$  with both  $V$  and  $V_1$  Jordan domains and a pair of homeomorphisms  $(\phi, \psi)$  from  $\overline{W}$  to  $\overline{V}$  such that  $\psi$  is isotopic to  $\phi$  rel  $\mathcal{P} \cup \partial W$ ,  $\psi(\theta(W_1)) = V_1$  and  $\phi \circ G_1 \circ \psi^{-1} = g_1$  on  $\overline{V_1}$ . Extend  $(\phi, \psi)$  to homeomorphisms of  $\widehat{\mathbb{C}}$  such that they coincide with each other outside of  $W$ . Let  $G_2 = \phi \circ G_1 \circ \psi^{-1}$ . Then  $(G_2, \phi(\beta))$  is Thurston equivalent to  $(F, \beta)$  and  $G_2|_{V_1} = g_1$  is a polynomial-like map. By Straightening Theorem (refer to [9] Theorem 1), there is a quasiconformal map  $h$  of  $\widehat{\mathbb{C}}$  such that for  $G := h \circ G_2 \circ h^{-1}$ ,  $G|_{h(V_1)}$  is a polynomial  $g$ . Therefore  $(F, \beta)$  is the folding of the polynomial  $g$ . The uniqueness of  $g$  comes from Thurston Theorem.

This argument also works for type B. We omit its proof.  $\square$

Combining Theorems 8.1, 8.2 and 8.3, we obtain

**Corollary 8.4.** *Let  $(F, \beta)$  be a folding of a polynomial  $g$  (or a pair of polynomials  $(g_1, g_2)$ ). Suppose that  $\{\beta\}$  is not a Thurston obstruction.*

(a) *If  $d(F, \beta) < m(F, \beta)$ , then  $F$  has no Thurston obstruction.*

(b) *Suppose that there exist an integer  $p \geq 1$  and a finite tree  $T \subset \widehat{\mathbb{C}} \setminus \beta$  whose vertices are contained in  $\mathcal{P}_F$  such that  $F^p(T)$  is contained in  $T$  homotopically,  $F^p$  is injective on  $T$  and  $F^{-p}(T)$  has a component homotopic to  $\beta$  rel  $\mathcal{P}_F$ . Then  $F$  has no Thurston obstruction.*

### 8.3 Proof of Theorems 8.1 and 8.2

Let  $(F, \beta)$  be a folding map. For any two essential Jordan curves  $\gamma$  and  $\alpha$  in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_F$ , set  $k(\gamma, \alpha)$  to be their geometric intersection number. It is defined by

$$k(\gamma, \alpha) = \min\{\#(\delta \cap \alpha)\},$$

where the minimum is taken over all the choices of  $\delta$  in the homotopy class of  $\gamma$ . By definition  $k(\gamma, \alpha) = 0$  if  $\gamma$  is homotopic to  $\alpha$  rel  $\mathcal{P}_F$ .

**Lemma 8.5.** *Let  $\Gamma$  be an irreducible multicurve of  $F$ . Then either  $k(\gamma, \beta) \neq 0$  for all  $\gamma \in \Gamma$  or  $k(\gamma, \beta) = 0$  for all  $\gamma \in \Gamma$ .*

*Proof.* Suppose that  $k(\gamma, \beta) = 0$  for some  $\gamma \in \Gamma$ . For any curve  $\alpha \in \Gamma$ , since  $\Gamma$  is irreducible,  $\alpha$  is homotopic to a component of  $F^{-n}(\gamma)$  rel  $\mathcal{P}_F$  for some  $n \geq 0$ . Let  $\delta$  be a Jordan curve in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_F$  homotopic to  $\gamma$  rel  $\mathcal{P}_F$  such that it is disjoint from  $\beta$ , then  $\alpha$  is homotopic to a component of  $F^{-n}(\delta)$  rel  $\mathcal{P}_F$ , which is disjoint from  $F^{-n}(\beta)$ . Thus  $k(\alpha, \beta) = 0$  since  $F^{-n}(\beta)$  contains a curve homotopic to  $\beta$  rel  $\mathcal{P}_F$ .  $\square$

**Lemma 8.6.** *Let  $\gamma$  and  $\alpha$  be essential Jordan curves in  $\widehat{\mathbb{C}} \setminus \mathcal{P}_F$  such that  $k(\gamma, \beta) \neq 0$ . Suppose that  $F^{-1}(\gamma)$  has a component  $\delta$  homotopic to  $\alpha$  rel  $\mathcal{P}_F$ . Then*

$$\deg(F : \delta \rightarrow \gamma) \geq \frac{m \cdot k(\alpha, \beta)}{k(\gamma, \beta)},$$

where  $m = m(F, \beta)$ .

*Proof.* Denote by  $d(\delta) = \deg(F : \delta \rightarrow \gamma)$ . We may assume that  $\#(\gamma \cap \beta) = k(\gamma, \beta)$ . Denote by  $\alpha_1, \dots, \alpha_m$  the components of  $F^{-1}(\beta)$ . Then

$$\# \left( \delta \cap \bigcup_{p=1}^m \alpha_p \right) = d(\delta) \cdot \#(\gamma \cap \beta) = d(\delta) \cdot k(\gamma, \beta).$$

On the other hand,

$$\# \left( \delta \cap \bigcup_{p=1}^m \alpha_p \right) = \sum_{p=1}^m \#(\delta \cap \alpha_p) \geq \sum_{p=1}^m k(\alpha, \beta) = m \cdot k(\alpha, \beta).$$

Combining the above two inequalities we get the lemma.  $\square$

**Lemma 8.7.** *Suppose that  $\Gamma = \{\gamma_1, \dots, \gamma_n\}$  is an irreducible multicurve of  $F$  such that  $k_i = k(\gamma_i, \beta) \neq 0$ . Let  $M_\Gamma = (a_{ij})$  be the transition matrix of  $\Gamma$ . Then*

$$a_{ij} \leq \frac{d_0 k_j^2}{m k_i^2},$$

where  $d_0 = d(F, \beta)$  and  $m = m(F, \beta)$ .

*Proof.* We may assume that  $\#(\gamma_i \cap \beta) = k(\gamma_i, \beta) = k_i$  for any  $\gamma_i \in \Gamma$ . Fix a pair  $(i, j)$ . If  $F^{-1}(\gamma_j)$  has no component homotopic to  $\gamma_i$ , then  $a_{ij} = 0$ . Now suppose that  $F^{-1}(\gamma_j)$  has  $n > 0$  components homotopic to  $\gamma_i$ . Denote them by  $\{\delta_s, s = 1, \dots, n\}$ . We claim that  $n \leq d_0 k_j / k_i$ .

Denote by  $U, V$  the two components of  $\widehat{\mathbb{C}} \setminus \beta$ . Denote by  $U_1, V_1$  the two disc components of  $\widehat{\mathbb{C}} \setminus F^{-1}(\beta)$  such that  $U_1, V_1$  are homotopic to  $U$  and  $V$ , respectively. Denote  $d_1 = \deg(F|_{U_1})$  and  $d_2 = \deg(F|_{V_1})$ . Then both  $U \cap \gamma_j$  and  $V \cap \gamma_j$  have exactly  $k_j/2$  components (notice that  $k_j = \#(\gamma_j \cap \beta)$  is an even number). It follows that  $U_1 \cap F^{-1}(\gamma_j)$  has exactly  $d_1 k_j/2$  components and  $V_1 \cap F^{-1}(\gamma_j)$  has exactly  $d_2 k_j/2$  components.

On the other hand, since both  $\partial U_1$  and  $\partial V_1$  are homotopic to  $\beta$  rel  $\mathcal{P}_F$ , both  $U_1 \cap \delta_s$  and  $V_1 \cap \delta_s$  have at least  $k_i/2$  components for  $s = 1, \dots, n$ . It follows that

$$\begin{aligned} \frac{n k_i}{2} &\leq \# \left\{ \text{components of } U_1 \cap (\cup \delta_s) \right\} \\ &\leq \# \left\{ \text{components of } U_1 \cap F^{-1}(\gamma_j) \right\} = \frac{d_1 k_j}{2}. \end{aligned}$$

So  $n \leq d_1 k_j / k_i$ . We also have  $n \leq d_2 k_j / k_i$  if we replace  $U_1$  by  $V_1$  in the above inequality. Hence  $n \leq \min\{d_1, d_2\} k_j / k_i \leq d_0 k_j / k_i$ .

Now applying Lemma 8.6, we have

$$a_{ij} = \sum_{s=1}^n \frac{1}{\deg(F : \delta_s \rightarrow \gamma_j)} \leq \sum_{s=1}^n \frac{k_j}{mk_i} = n \frac{k_j}{mk_i} \leq d_0 \frac{k_j}{k_i} \frac{k_j}{mk_i} = \frac{d_0 k_j^2}{mk_i^2}.$$

This proves the lemma.  $\square$

*Proof of Theorem 8.1.* Denote  $m = m(F, \beta)$ ,  $d_0 = d(F, \beta)$  and  $p = \#\mathcal{P}_F$ . By the condition  $d_0 < m$ , there is an integer  $N \geq 1$  such that  $(p-3)d_0^N < m^N$ .

Now we consider the folding map  $(F^N, \beta)$ . Note that  $\#\mathcal{P}_{F^N} = p$ ,  $m(F^N, \beta) = m^N$  and  $d(F^N, \beta) = d_0^N$ . Let  $\Gamma = \{\gamma_1, \dots, \gamma_n\}$  be an irreducible multicurve of  $F^N$ . If  $k(\gamma_i, \beta) = 0$  for some  $\gamma_i \in \Gamma$ , then  $k(\gamma_j, \beta) = 0$  for all  $\gamma_j \in \Gamma$  by Lemma 8.5. So  $\Gamma$  is homotopic rel  $\mathcal{P}_F$  to a multicurve disjoint from  $\beta$ . Thus it is not a Thurston obstruction by the conditions (a) and (b).

Now we assume that  $k_i = k(\gamma_i, \beta) \neq 0$  for all  $\gamma_i \in \Gamma$ . Set the vector  $\mathbf{v} = (v_i)$  with  $v_i = 1/k_i^2$ , then by Lemma 8.7

$$(M_\Gamma \mathbf{v})_i = \sum_{j=1}^n a_{ij} v_j \leq \sum_{j=1}^n \frac{d_0^N k_j^2}{m^N k_i^2} \cdot \frac{1}{k_j^2} = \frac{n d_0^N}{m^N k_i^2}.$$

Notice that  $n = |\Gamma| \leq \#\mathcal{P}_{F^N} - 3 = p - 3$ . Therefore

$$(M_\Gamma \mathbf{v})_i \leq \frac{(p-3)d_0^N}{m^N k_i^2} < \frac{1}{k_i^2} = v_i.$$

It follows that  $M_\Gamma \mathbf{v} < \mathbf{v}$  and hence  $\lambda(\Gamma) < 1$  (refer to Lemma A.1 in [6]). Thus  $F^N$  has no Thurston obstruction by Lemma 2.2. This implies that  $F$  has no Thurston obstruction.  $\square$

*Proof of Theorem 8.2.* Denote  $m = m(F, \beta)$ . Let  $\Gamma = \{\gamma_1, \dots, \gamma_n\}$  be an irreducible multicurve of  $F$ . If  $k(\gamma_i, \beta) = 0$  for some  $\gamma_i \in \Gamma$ , then  $k(\gamma_j, \beta) = 0$  for all  $\gamma_j \in \Gamma$  by Lemma 8.5. So  $\Gamma$  is homotopic rel  $\mathcal{P}_F$  to a multicurve disjoint from  $\beta$ . Thus it is not a Thurston obstruction by the conditions (a) and (b).

Now we assume that  $k(\gamma_i, \beta) \neq 0$  for all  $\gamma_i \in \Gamma$ . Denote by  $k(\gamma_i, T)$  the geometric intersection number.

*Case 1.*  $k(\gamma_i, T) = 0$  for some curve  $\gamma_i \in \Gamma$ . Assume that  $\gamma_i$  is disjoint from  $T$ . By condition (c'),  $F^{-p}(T)$  has a component homotopic to  $\beta$  rel  $\mathcal{P}_F$ . Thus  $k(\delta, \beta) = 0$  for all the components  $\delta$  of  $F^{-p}(\gamma_i)$ . But  $\Gamma$  is irreducible, so  $k(\gamma_j, \beta) = 0$  for some  $\gamma_j \in \Gamma$ . It is a contradiction.

*Case 2.*  $k(\gamma_i, T) \neq 0$  for all curves  $\gamma_i \in \Gamma$ . We may assume that  $p = 1$  and  $F(T) \subset T$ . We also assume that  $\#(\gamma_i \cap T) = k(\gamma_i, T)$  for  $\gamma_i \in \Gamma$ . Let  $\delta_s$  ( $s = 1, \dots, n$ ) be all the components of  $F^{-1}(\gamma_j)$  homotopic to a curve in  $\Gamma$ , then  $F : (\cup \delta_s) \cap T \rightarrow \gamma_j \cap T$  is injective since  $F(T) \subset T$  and  $F|_T$  is injective. So

$$\sum_{s=1}^n \#(\delta_s \cap T) \leq \#(\gamma_j \cap T) = k(\gamma_j, T).$$

Therefore if  $\gamma_i$  is homotopic to a curve in  $F^{-1}(\gamma_j)$ , then  $k(\gamma_i, T) \leq k(\gamma_j, T)$ . Since  $\Gamma$  is irreducible, we have  $k(\gamma_i, T)$  is a constant for all  $\gamma_i \in \Gamma$  and  $F^{-1}(\gamma_j)$  has exactly one component homotopic to a curve in  $\Gamma$ . Relabel the index such that  $F^{-1}(\gamma_{j+1})$  has a curve  $\delta_j$  homotopic to  $\gamma_j$  for  $j = 1, \dots, n$  (set  $\gamma_{n+1} = \gamma_1$ ). Let  $M_\Gamma = (a_{ij})$  be the transition matrix of  $\Gamma$ . Then

$$a_{j,j+1} = \frac{1}{\deg(F : \delta_j \rightarrow \gamma_{j+1})}$$

for  $j = 1, \dots, n$  (set  $a_{n,n+1} = a_{n,1}$ ), and  $a_{ij} = 0$  otherwise. By Lemma 8.6, we have

$$\deg(F : \delta_j \rightarrow \gamma_{j+1}) \geq \frac{mk_j}{k_{j+1}}.$$

So  $a_{j,j+1} \leq k_{j+1}/(mk_j)$ . Set the vector  $\mathbf{v} = (1/k_i)$ , then  $M_\Gamma \mathbf{v} \leq (1/m)\mathbf{v}$  and hence  $\lambda(\Gamma) \leq 1/m < 1$  (refer to Lemma A.1 in [6]). Applying Lemma 2.2 again we conclude that  $F$  has no Thurston obstruction.  $\square$

## 8.4 Sierpinski maps

A connected compact subset  $E \subset \widehat{\mathbb{C}}$  is called a **Sierpinski carpet** if it is locally connected, nowhere dense and its complementary components are Jordan domains with pairwise disjoint closures.

A rational map is called a **Sierpinski map** if its Julia set is a Sierpinski carpet. A polynomial is called **Sierpinski type** if its Julia set is connected and locally connected, it has at least two bounded Fatou domains and any two bounded Fatou domains have disjoint closures.

**Theorem 8.8.** *Let  $f$  be a post-critically finite rational map. Suppose that  $(f, \beta)$  is a folding of a Sierpinski type polynomial  $g$ . Then  $f$  is a Sierpinski map.*

*Proof.* Note that  $\{\beta\}$  is a Cantor multicurve of  $f$ . Applying Theorem 4.1, there exists an annulus  $A \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$  homotopic to  $\beta$  rel  $\mathcal{P}_f$ , such that  $f : f^{-1}(A) \rightarrow A$  is an exact annular system.

Denote by  $B_1, B_2$  the two components of  $\widehat{\mathbb{C}} \setminus A$ , then they are components of  $f^{-1}(B_1 \cup B_2)$ . By rearranging the index, we may assume that  $f(B_1) = f(B_2) = B_1$ . Then there exists a Jordan curve  $\beta_0$  essentially contained in  $A$  such that  $U_1 \Subset U_0$ , where  $U_0$  is the domain enclosed by  $\beta_0$  and containing  $B_1$  and  $U_1$  is the pre-image of  $f^{-1}(U_0)$  containing  $B_1$ . Since  $\beta_0$  is homotopic to  $\beta$  rel  $\mathcal{P}_f$ , the polynomial-like map  $g_1 = f|_{U_1} : U_1 \rightarrow U_0$  is quasiconformally conjugate to a restriction of the polynomial  $g$ .

Each periodic Fatou domain of  $f$  is contained in  $B_1$ . Thus it is a periodic Fatou domain of the polynomial-like map  $g_1$ . Since the polynomial  $g$  is of Sierpinski type, every periodic Fatou domain of  $f$  is a Jordan domain. Since  $f$  is hyperbolic, every Fatou domain of  $f$  is a Jordan domain.

Any two Fatou domains of  $f$  are either contained in the same component of  $f^{-n}(B_1)$  for some integer  $n \geq 0$ , or separated by a component of  $f^{-m}(A)$  for some integer  $m \geq 0$  and hence have disjoint closures. In the former case they have disjoint closures since  $g$  is of Sierpinski type. So  $f$  is a Sierpinski map.  $\square$



One may construct Sierpinski maps from the above theorem. Refer to Example 3 in the next section.

## 8.5 Examples

Let  $g$  be a monic post-critically finite polynomial with  $\#\mathcal{P}_g \geq 3$  and  $\deg g = d_1$ . We want to construct a folding map  $(F, \beta)$  such that it is the folding of  $g$ . Denote by:

$$\mathbb{D} = \{z : |z| < 1\} \text{ and } \beta = \partial\mathbb{D},$$

$$\Delta = \{z : |z| > 2\} \cup \{\infty\} \text{ and } \beta_* = \partial\Delta,$$

$$A = \{z : 1 \leq |z| \leq 2\},$$

$$\rho : \mathbb{D} \rightarrow \mathbb{C}, \text{ a homeomorphism defined by } \rho(z) = -\frac{z}{|z|} \log(1 - |z|),$$

$$\sigma : \Delta \rightarrow \mathbb{C}, \text{ a homeomorphism defined by } \sigma(z) = \rho(2/z).$$

Define  $G_1 : \mathbb{D} \rightarrow \mathbb{D}$  by  $G_1 = \rho^{-1} \circ g \circ \rho$ . Then  $G_1$  can be extended continuously to the boundary with  $G_1(z) = z^{d_1}$  on  $\beta$ .

Define  $G_2 : \Delta \rightarrow \mathbb{D}$  to be a branched covering with  $\deg G_2 = d_2 \geq 2$  such that it can be continuously extended to a covering from  $\beta_*$  to  $\beta$ .

Define  $G_3 : A \rightarrow A \cup \Delta$  to be a branched covering such that its boundary value coincides with  $G_1$  on  $\beta$ , and  $G_2$  on  $\beta_*$ , and its critical values are contained in  $\Delta$ .

Define  $F = G_1$  on  $\mathbb{D}$ ,  $F = G_2$  on  $\Delta$  and  $F = G_3$  on  $A$ . Then  $F$  is a branched covering of  $\widehat{\mathbb{C}}$ . We always assume that both  $G_2(\Omega_F \cap \Delta)$  and  $G_2 \circ G_3(\Omega_F \cap A)$  are eventually periodic points of  $G_1$ . Then  $F$  is post-critically finite. It follows that  $(F, \beta)$  is the folding of the polynomial  $g$ .

**Example 1.** Let  $g = Q_c^2$ , where  $Q_c(z) = z^2 + c$  is the airplane quadratic polynomial, i.e. the parameter  $c$  is the unique real solution of the equation  $(c^2 + c)^2 + c = 0$ . Then  $Q_c$  has a super-attracting cycle with period  $p = 3$  on the real axis.

Let  $x_0 < 0$  be the  $\alpha$ -fixed point of  $Q_c$ . Let  $Q_c^{-1}(-x_0) = \{x_1, -x_1\}$  with  $x_1 < 0$ . Then

$$c < x_1 < x_0 < 0 < -x_0 < c^2 + c < -x_1.$$

Denote by  $R(\theta)$  the external ray of  $Q_c$  of angle  $2\pi\theta$ . Then both  $R(1/3)$  and  $R(2/3)$  land on the  $\alpha$ -fixed point  $x_0$ . Denote by  $L_0 = R(1/3) \cup \{x_0\} \cup R(2/3)$ . Then  $g^{-1}(L_0)$  has 4 lines:  $L_0, -x_0 \in L_1, x_1 \in L_2$  and  $-x_1 \in L_3$ . The line segments  $S_i = \rho^{-1}(L_i)$  in  $\mathbb{D}$  land on  $\beta$  at  $(e^{2\pi\theta_i}, e^{2\pi\varphi_i})$  with  $(\theta_i, \varphi_i) = (1/3, 2/3), (1/6, 5/6), (5/12, 7/12)$  and  $(1/12, 11/12)$ , respectively.

Define  $G_1 : \mathbb{D} \rightarrow \mathbb{D}$  by  $G_1 = \rho^{-1} \circ g \circ \rho$ . Then  $G_1^{-1}(S_0) = \bigcup_{i=0}^3 S_i$ .

Define  $G_2 : \Delta \rightarrow \mathbb{D}$  by  $G_2 = \rho^{-1} \circ g \circ \sigma$ , then  $G_2|_{\beta_*}(z) = (2/z)^4$ . Denote  $E_i = \sigma^{-1}(L_i)$ . Then  $G_2^{-1}(S_0) = \bigcup_{i=0}^3 E_i$ . Note that the line segments  $E_i$  on  $\Delta$  land on  $\beta_*$  at  $(2e^{2\pi\theta_i}, 2e^{2\pi\varphi_i})$ .

Set  $I_i = \{re^{2\pi\theta_i}, r \in [1, 2]\}$  and  $J_i = \{re^{2\pi\varphi_i}, r \in [1, 2]\}$  for  $i = 0, 1, 2, 3$ . Then

$\gamma_i = S_i \cup E_i \cup I_i \cup J_i$  are Jordan curves. Define  $G_3$  on  $I_i \cup J_i$  such that  $G_3 : I_i \rightarrow I_i \cup E_i \cup J_i$  are homeomorphisms and  $G_3 : J_i \rightarrow I_i \cup E_i \cup J_i$  are also homeomorphisms. Extend  $G_3$  continuously in each of the 8 remaining discs in  $A$  as a branched covering of degree 2, such that the unique critical point is mapped to  $y_1 = \sigma^{-1}(c)$  or  $y_0 = \sigma^{-1}(0)$ .

Define  $F_1 = G_1$  on  $\mathbb{D}$ ,  $F_1 = G_2$  on  $\Delta$  and  $F_1 = G_3$  on  $A$ . Then  $F_1$  is a branched covering of  $\widehat{\mathbb{C}}$  with post-critical set  $\mathcal{P}_{F_1} = \rho^{-1}(\mathcal{P}_g) \cup \{y_0, y_1\}$ .

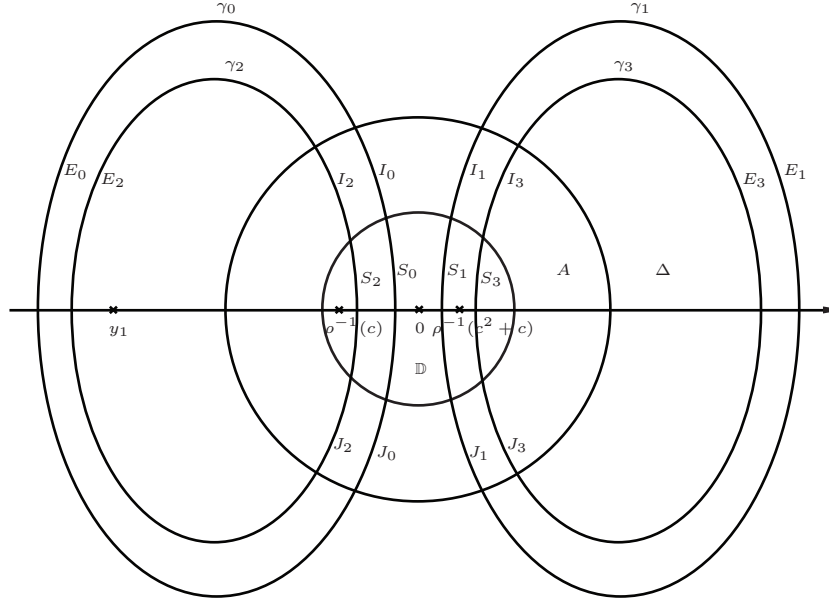


Figure 2. Construction of  $F_1$

**Example 2.** Let  $U$  be the Fatou domain of  $Q_c$  containing  $c$ . Let  $x_2 \in (c, x_1)$  be the right intersection point of  $\partial U$  with real axis. Set  $y_2 = \sigma^{-1}(x_2)$ . Then there is a homeomorphism  $h$  of  $\widehat{\mathbb{C}}$  such that  $h$  is the identity outside of  $\Delta$ ,  $h(y_1) = y_1$  and  $h(y_0) = y_2$ . Define  $F_2 = h \circ F_1$ . Then  $\mathcal{P}_{F_2} = \rho^{-1}(\mathcal{P}_g) \cup \{y_2, y_1\}$ .

Note that both  $(F_1, \beta)$  and  $(F_2, \beta)$  satisfy the conditions (a) and (b) in Theorems 8.1 and 8.2. Since  $m(F_i, \beta) = 2$  and  $d(F_i, \beta) = 4$  ( $i = 1, 2$ ),  $(F_i, \beta)$  do not satisfy the condition (c) in Theorem 8.1.

**Proposition 8.9.** *The folding map  $(F_1, \beta)$  has a Thurston obstruction, whereas the folding map  $(F_2, \beta)$  has no Thurston obstruction.*

*Proof.* The curve  $\gamma_0$  has 4 pre-images  $\gamma_i$  ( $i = 0, \dots, 3$ ) with  $\deg(F_1 : \gamma_i \rightarrow \gamma_0) = 2$ . The curve  $\gamma_3$  is null-homotopic and  $\gamma_1$  is peripheral. Both  $\gamma_0$  and  $\gamma_2$  are essential and homotopic to each other rel  $\mathcal{P}_{F_1}$ . Set  $\Gamma = \{\gamma_0\}$ . Then  $\Gamma$  is a stable multicurve with  $\lambda(\Gamma) = 1/2 + 1/2 = 1$ . So it is a Thurston obstruction.

Let  $T_0 = [y_1, y_2]$ . Then  $F_2(\Omega_{F_2} \cap A) = \{y_1, y_2\} \subset T_0$ . So  $F_2^{-1}(T_0)$  has a component  $K$  essentially contained in  $A$ . Thus  $K$  is homotopic to  $\beta$  rel  $\mathcal{P}_{F_2}$ . Set  $T_1 := F_2(T_0) = \rho^{-1} \circ g([c, x_2])$ . Then the line segment  $T_1$  eventually maps to a  $p$ -periodic interval  $T$  and  $F_2^p$  is injective on  $T$ . Applying Theorem 8.2, the map  $F_2$  has no Thurston obstructions.  $\square$

**Example 3.** Let  $\beta_i$  ( $i = 1, \dots, 6$ ) be disjoint Jordan curves essentially contained in the interior of  $A$  such that  $\mathbb{D} \subset D_1 \subset \dots \subset D_6$ , where  $D_i$  is the bounded domain enclosed by  $\beta_i$ . Denote by  $A_i = D_i \setminus \overline{D_{i-1}}$  (set  $D_0 = \mathbb{D}$  and  $D_7 = \widehat{\mathbb{C}} \setminus \overline{\Delta}$ ) for  $i = 1, \dots, 7$ .

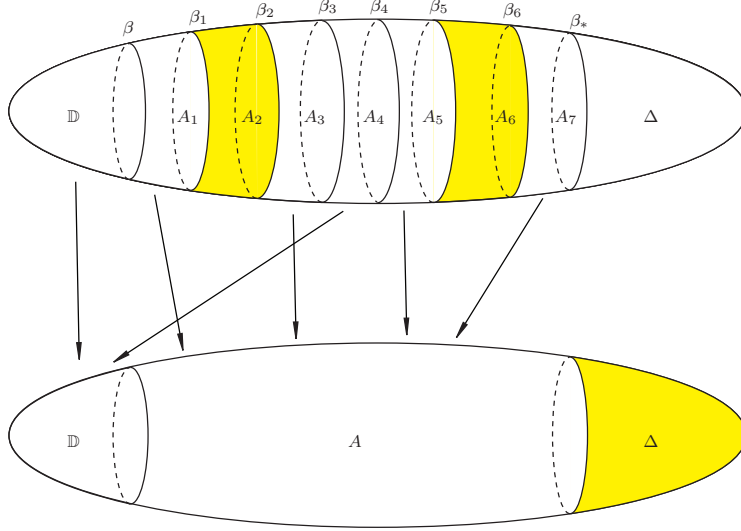


Figure 3. Construction of  $F_3$

Let  $Q_c$  be the airplane quadratic polynomial. It is a Sierpinski type polynomial. Denote by  $\mathcal{O}$  the grand orbits of its super-attracting cycles. Define a branched covering  $F_3$  of  $\widehat{\mathbb{C}}$  by the following:

- $F_3 : \mathbb{D} \rightarrow \mathbb{D}$  is defined by  $F_3 = \rho^{-1} \circ Q_c \circ \rho$ .
- $F_3 : \Delta \rightarrow \mathbb{D}$  is defined by  $F_3(z) = (2/z)^4$ .
- $F_3 : A_1, A_3, A_5, A_7 \rightarrow A$  are coverings with degree 2, 8, 16, 4, respectively such that they can be continuously extended to covering maps on the boundary with  $\beta_3, \beta_4$  mapping to  $\beta$  and coincide with the previous defined boundary maps on  $\beta \cup \beta_*$ .
- $F_3 : A_4 \rightarrow \mathbb{D}$  is a branched covering such that its boundary map coincides with the previous defined boundary maps and its critical values map into  $\rho^{-1}(\mathcal{O})$ . Clearly its degree is 24.
- $F_3 : A_2, A_6 \rightarrow \Delta$ , are branched coverings such that their boundary maps coincide with the previous defined boundary maps and their critical values map into  $(F_3|_{\Delta})^{-1} \circ \rho^{-1}(\mathcal{O})$ . Clearly their degrees are 10 and 20, respectively.

Now  $(F_3, \beta)$  is a folding of the polynomial  $Q_c$  and every critical point of  $F_3$  is eventually super-attracting. Note that  $m(F_3, \beta) = 4$  and  $d(F_3, \beta) = 2$ . Thus  $F_3$  has no Thurston obstruction by Theorem 8.1 and hence is Thurston equivalent to a hyperbolic rational map  $f$  (note that  $\#\mathcal{P}_{F_3} \geq 5$  and hence its orbifold is hyperbolic) by Thurston Theorem. By Theorem 8.8,  $f$  is a Sierpinski map.

**Proposition 8.10.** *The folding map  $(F_3, \beta)$  does not satisfy the condition (c') in Theorem 8.2.*

*Proof.* We only need to prove that there is no finite tree  $T \subset \mathbb{D}$  whose vertices are contained in  $\mathcal{P}_{F_3}$  such that  $F_3^p(T)$  is contained in  $T$  homotopically for some integer  $p \geq 1$  and  $F_3^p$  is injective on  $T$ . Note that each point in  $\mathcal{P}_{F_3} \cap \mathbb{D}$  eventually maps to a critical cycle. Since restricted on  $\mathbb{D}$ ,  $F_3$  is topologically conjugated to the polynomial  $Q_c$ , we only need to prove that there is no finite tree  $T \subset \widehat{\mathbb{C}}$  whose vertices are contained in the grand orbits of super-attracting cycles of  $Q_c$ , such that  $Q_c^p(T)$  is contained in  $T$  homotopically and  $Q_c^p$  is injective on  $T$ .

In fact, we can prove a stronger statement: For any Sierpinski polynomial  $g$  and any two centers  $z_1, z_2$  of Fatou domains of  $g$ , there exists no arc  $\delta$  connecting them such that  $g^p$  is injective on  $\delta$  for some integer  $p \geq 1$  and  $g^p(\delta)$  is homotopic to  $\delta$  rel  $\mathcal{P}_g \cup \{z_1, z_2\}$ . Otherwise, by taking pre-images consecutively and apply Shrinking Lemma in [16], we can show that these two Fatou domain have common boundary points. It contradicts the condition that  $g$  is of Sierpinski type.  $\square$

## A Rees-Shishikura's semi-conjugacy

Let  $F$  be a formal mating of two post-critically finite polynomials. Suppose that  $F$  is Thurston equivalent to a rational map  $f$ . Then there is a semi-conjugacy from  $F$  to  $f$ . This result was obtained by M. Rees if both polynomials are hyperbolic [21], and proved by Shishikura in general [22]. The same result is still true for general post-critically finite branched coverings. Here we provide a statement with a general form.

**Theorem A.1.** *Let  $F : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a post-critically finite branched covering which is Thurston equivalent to a rational map  $f$  through a pair of homeomorphisms  $(\psi_0, \psi_1)$  of  $\widehat{\mathbb{C}}$ . Suppose that  $F$  is holomorphic in a neighborhood of the critical cycles of  $F$ . Then there exist a neighborhood  $U$  of the critical cycles of  $F$  and a sequence of homeomorphisms  $\{\phi_n\}$  ( $n \geq 0$ ) of  $\widehat{\mathbb{C}}$  homotopic to  $\psi_0$  rel  $\mathcal{P}_F$ , such that  $\phi_n|_U$  is holomorphic,  $\phi_n|_U = \phi_0|_U$  and  $f \circ \phi_{n+1} = \phi_n \circ F$ . The sequence  $\{\phi_n\}$  converges uniformly to a continuous map  $h : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ . Moreover, the following statements hold:*

- (1)  $h \circ F = f \circ h$ .
- (2)  $h$  is surjective.
- (3)  $h^{-1}(w)$  is a single point for  $w \in \mathcal{F}_f$  and  $h^{-1}(w)$  is either a single point or a full continuum for  $w \in \mathcal{J}_f$ .
- (4) For points  $x, y \in \widehat{\mathbb{C}}$  with  $f(x) = y$ ,  $h^{-1}(x)$  is a component of  $F^{-1}(h^{-1}(y))$ . Moreover,  $\deg F|_{h^{-1}(x)} = \deg_x f$ .
- (5)  $h^{-1}(E)$  is a continuum if  $E \subset \widehat{\mathbb{C}}$  is a continuum.
- (6)  $h(F^{-1}(E)) = \widehat{f^{-1}(h(E))}$  for any  $E \subset \widehat{\mathbb{C}}$ .
- (7)  $F^{-1}(\widehat{E}) = \widehat{F^{-1}(E)}$  for any  $E \subset \widehat{\mathbb{C}}$ , where  $\widehat{E} = h^{-1}(h(E))$ .

One may also refer to [7] for a detailed account in a generalized form. The crucial part of theorem is the construction of the homotopy  $(\phi_0, \phi_1)$  rel a neighborhood  $U$  of critical cycles and the convergence of the sequence  $\{\phi_n\}$ . The other statements are directly deduced. The statements (5)-(7) is used in this paper, so we add them in the theorem and provide a proof here.

*Proof of (5)-(7).* (5). Suppose that  $E \subset \widehat{\mathbb{C}}$  is a connected closed subset. The closeness of  $h^{-1}(E)$  is easy to see. Now suppose that  $h^{-1}(E)$  is not connected, i.e., there are open sets  $U_1, U_2$  in  $\widehat{\mathbb{C}}$  such that  $h^{-1}(E) \subset U_1 \cup U_2$ ,  $U_1 \cap U_2 = \emptyset$  and both  $U_1$  and  $U_2$  intersect with  $h^{-1}(E)$ . Then  $K := h(\widehat{\mathbb{C}} \setminus (U_1 \cup U_2))$  is a compact set disjoint from  $E$ . Since  $E$  is connected, there is a connected neighborhood  $V$  of  $E$  such that  $\overline{V} \cap K = \emptyset$ . Since  $\{\phi_n\}$  converges uniformly to  $h$ , there exists an integer  $n > 0$  such that

$$d(h, \phi_n) = \sup_{z \in \widehat{\mathbb{C}}} d(h(z), \phi_n(z)) < \min \left\{ d(E, \partial V), d(\overline{V}, K) \right\},$$

where  $d(\cdot, \cdot)$  denotes the spherical distance. Then it follows that  $\phi_n(\widehat{\mathbb{C}} \setminus (U_1 \cup U_2)) \cap \overline{V} = \emptyset$ , hence  $\phi_n^{-1}(V) \subset U_1 \cup U_2$ . On the other hand, since  $V \supset E$ , both  $U_1$  and  $U_2$  intersect with  $\phi_n^{-1}(V)$ . This contradicts the fact that  $\phi_n^{-1}(V)$  is connected.

(6). From  $f \circ h(F^{-1}(E)) = h \circ F(F^{-1}(E)) = h(E)$ , we have  $h(F^{-1}(E)) \subset f^{-1}(h(E))$ . Conversely, for any point  $w \in f^{-1}(h(E))$ ,  $f(w) \in h(E)$ . So there is a point  $z_0 \in E$  such that  $f(w) = h(z_0)$ . By (5), the map

$$F : h^{-1}(w) \rightarrow h^{-1}(f(w))$$

is surjective. Noticing that  $z_0 \in h^{-1}(f(w))$ , there is a point  $z_1 \in h^{-1}(w)$  such that  $F(z_1) = z_0$ . So  $w = h(z_1) \in h(F^{-1}(z_0)) \subset h(F^{-1}(E))$ . Therefore,  $f^{-1}(h(E)) \subset h(F^{-1}(E))$ .

(7).  $F^{-1}(\widehat{E}) = F^{-1}(h^{-1}(h(E))) = h^{-1}(f^{-1}(h(E)))$ . From (6), we obtain

$$F^{-1}(\widehat{E}) = h^{-1}(h(F^{-1}(E))) = \widehat{F^{-1}(E)}.$$

□

## B Buff's example

**Example.** Denote by  $U = \{z : 1 < |z| < r_0\}$  with  $r_0 > 2$ . Define a spiral in  $U$  by:

$$\delta = \left\{ \rho e^{i\theta} : \frac{r_0}{2} \leq \rho < r_0, \theta = \frac{1}{r_0 - \rho} \right\}.$$

Set  $A_1 = U \setminus \delta$ . Then  $A_1$  is an annulus with modulus  $\text{mod}(A_1) < \log r_0 / (2\pi)$ . Pick an integer  $d > 2$  such that  $d \text{mod}(A_1) > \log r_0 / \pi$ . Set  $r_1 > 1$  be the constant such that  $\log r_1 / (2\pi) = d \text{mod}(A_1)$ . Then  $r_1 > r_0^2$ . Set  $A = \{z : 1 < |z| < r_1\}$  and  $A_2 = h(A_1)$  with  $h(z) = r_1/z$ . Then  $A_2$  is disjoint from  $A_1$  and there is a holomorphic covering  $g$  of degree  $d$  from  $A_i$  ( $i = 1, 2$ ) to  $A$  such that  $g$  fixes the two components of  $\partial A$ . Then  $g : A_1 \cup A_2 \rightarrow A$  is an exact annular system.

**Theorem B.1.** *Let  $\mathcal{J}$  be the collection of the components of the Julia set of the exact annular system  $g : A_1 \cup A_2 \rightarrow A$ . With the topology induced by the corresponding linear system,  $\mathcal{J}$  has a dense subset whose elements are not locally connected.*

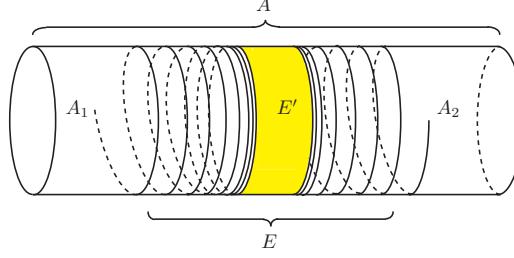


Figure 4. An annular system with spirals

Set  $B = \{\zeta, 0 < \text{Im } \zeta < \log r_1\}$ . Then  $\pi(\zeta) = \exp(\zeta) : B \rightarrow A$  is a universal covering. Denote by  $E_0 = A \setminus (A_1 \cup A_2)$ . For each component  $E_n$  of  $g^{-n}(E_0)$  ( $n \geq 0$ ) and any point  $z \in A \setminus E_n$ , denote by

$$\text{h-dist}(z, \partial A; E_n) = \inf \left\{ \text{diam}(\pi^{-1}(\gamma)) \right\},$$

where the infimum is taken over all the open arc in  $A \setminus E_n$  connecting  $z$  with  $\partial A$ , and  $\text{diam}(\pi^{-1}(\gamma))$  is the Euclidean diameter of a component of  $\pi^{-1}(\gamma)$ .

Denote by  $E'_0 = A \setminus (U \cup h(U)) = E_0 \setminus (\delta \cup h(\delta))$ . It is easy to check that for any constant  $M < \infty$ , there exists a constant  $\varepsilon > 0$  such that for any point  $z \in A \setminus E_0$ , if the Euclidean distance  $\text{dist}(z, E'_0) < \varepsilon$ , then  $\text{h-dist}(z, \partial A; E_0) > M$ .

For each component  $E_n$  of  $g^{-n}(E_0)$  and any  $k > n \geq 0$ , denote by  $V_k(E_n)$  the union of the two components of  $g^{-k}(A)$  whose closures meet  $E_n$ . Then  $\tilde{V}_k(E_n) := V_k(E_n) \cup E_n$  is an annulus with  $\tilde{V}_{k+1}(E_n) \subset \tilde{V}_k(E_n)$  and  $\bigcap_{k \geq n} \tilde{V}_k(E_n) = E_n$ .

By the above argument, we see that for any constant  $M < \infty$ , there is an integer  $k(M) \geq 1$  such that for any component  $K$  of  $\mathcal{J}_g \cap V_{k(M)}(E_0)$ , there exists a point  $z \in K$  such that  $\text{h-dist}(z, \partial A; E_0) > M$ . By Koëbe distortion theorem, we may prove the next lemma.

**Lemma B.2.** *For any component  $E_n$  of  $g^{-n}(E_0)$  ( $n \geq 0$ ) and any constant  $M < \infty$ , there is an integer  $k(M, E_n) > n$  such that for any component  $K$  of  $\mathcal{J}_g \cap V_{k(M, E_n)}(E_n)$ , there exists a point  $z \in K$  such that  $\text{h-dist}(z, \partial A; E_n) > M$ .*

*Proof of Theorem B.1.* For each integer  $m > 0$  and any component  $E_n$  of  $g^{-n}(E_0)$  ( $n \geq 1$ ), define  $\mathcal{N}(m, E_n)$  to be the sub-collection of  $\mathcal{J}$  such that  $K \in \mathcal{N}(m, E_n)$  if  $K \subset V_{k(m, E_n)}(E_n)$ . Then  $\mathcal{N}(m, E_n)$  is an open set in  $\mathcal{J}$ . Set  $\mathcal{N}(m)$  to be the union of  $\mathcal{N}(m, E_n)$  for all  $n \geq 1$  and all the components  $E_n$  of  $g^{-n}(E_0)$ . Then it is an open dense subset of  $\mathcal{J}$ . Thus  $\mathcal{N} = \bigcap_{m \geq 1} \mathcal{N}(m)$  is a dense subset of  $\mathcal{J}$  in Baire's category.

For each  $K \in \mathcal{N}$ , we want to show that  $K$  is not locally connected. Otherwise  $K$  is a Jordan curve and hence there is an constant  $M < \infty$  such that for any point  $z \in K$ , there are open arcs  $\delta_+(z)$  and  $\delta_-(z)$  in  $A \setminus K$  connecting  $z$  with the two components of  $\partial A$ , respectively, such that  $\text{diam}(\pi^{-1}(\delta_{\pm}(z))) < M$ .

Fix an integer  $m > M$ . Since  $K \in \mathcal{N}(m)$ , there exist an integer  $n \geq 0$  and a component  $E_n$  of  $g^{-n}(E_0)$  such that  $K \in \mathcal{N}(m, E_n)$ . Thus there is a point  $z \in K$  such that  $\text{h-dist}(z, \partial A; E_n) > m > M$ . It contradicts that  $\text{diam}(\pi^{-1}(\delta_{\pm}(z))) < M$ .  $\square$



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